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Phase segregation dynamics for the Blume–Capel model with Kac interaction

R. Marra[a,*], M. Mourragui[b]

[a] Dipartimento di Fisica e Unitá INFM, Università di Roma Tor Vergata, Via della Ricerca Scientifica, 00133 Roma, Italy

[b] UPRESA 6085, Université de Rouen, 76821 Mont Saint Aignan, France

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Abstract

We consider the Glauber and Kawasaki dynamics for the Blume–Capel spin model with weak long-range interaction on the infinite lattice: a ferromagnetic d-dimensional lattice system with the spin variable σ taking values in $\{-1,0,1\}$ and pair Kac potential $\gamma^d J(\gamma(|i-j|))$, $\gamma>0$, $i,j\in\mathbb{Z}^d$. The Kawasaki dynamics conserves the empirical averages of σ and σ^2 corresponding to local magnetization and local concentration. We study the behaviour of the system under the Kawasaki dynamics on the spatial scale γ^{-1} and time scale γ^{-2}. We prove that the empirical averages converge in the limit $\gamma\to 0$ to the solutions of two coupled equations, which are in the form of the flux gradient for the energy functional. In the case of the Glauber dynamics we still scale the space as γ^{-1} but look at finite time and prove in the limit of vanishing γ the law of large number for the empirical fields. The limiting fields are solutions of two coupled nonlocal equations. Finally, we consider a nongradient dynamics which conserves only the magnetization and get a hydrodynamic equation for it in the diffusive limit which is again in the form of the flux gradient for a suitable energy functional.

Keywords: Interacting particle and spin systems; Kac potential; Hydrodynamic limits; Phase segregation

1. Introduction

We consider particle models which are dynamical versions of lattice gases with Kac potentials. The Kac potentials are functions $J_\gamma(r)$, $r\in\mathbb{R}^d$, $\gamma>0$, such that $J_\gamma(r)=\gamma^d J(\gamma r)$, where J is a smooth function of compact support. They have been introduced to describe particle (or spin) systems with weak long-range interaction between two particles (Kac et al., 1963). In the limit $\gamma\to 0$ the van der Waals theory of phase transition holds exactly for these models (Lebowitz and Penrose, 1963). Here we propose to consider a Blume–Capel model with Kac interaction that we call Kac–Blume–Capel (KBC) model. The Blume–Capel model is a spin system on the lattice with nearest-neighbour interactions such that the spin variable can assume three values: $-1,0,1$. It has been introduced originally to study the He^3–He^4 phase transition

* Corresponding author.

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(Blume, 1966; Capel, 1966). The structure of the phase diagram at low temperature for this model is well understood in terms of the Pirogov–Sinai theory (Bricmont and Slawny, 1989).

The KBC model is defined by the formal Hamiltonian

$$H_\gamma(\sigma) = \frac{1}{2} \sum_{i,j\in\mathbb{Z}^d} J_\gamma(i-j)(\sigma(i)-\sigma(j))^2 - h_1 \sum_{i\in\mathbb{Z}^d} \sigma(i) - h_2 \sum_{i\in\in\mathbb{Z}^d} \sigma^2(i), \tag{1.1}$$

where h_1 and h_2 are two real parameters. In Section 2 we provide for this model an analogue of the Lebowitz–Penrose theorem, showing that in the limit $\gamma \to 0$ the mean field theory of the Blume–Capel model (Blume et al., 1971) becomes exact. The equilibrium properties and the phase diagram of the model in the limit $\gamma \to 0$ are very interesting. There are two order parameters characterizing the equilibrium Gibbs measure: the magnetization m, the mean value of the spin, and the concentration ϕ, the mean value of the square of the spin. For inverse temperature β not larger than a critical value β_c there is a unique Gibbs measure which is indeed a Bernoulli measure (as usual for these mean field theories), while for temperatures sufficiently small (and suitable values of the parameters h_1 and h_2) the Gibbs measure is a superposition of Bernoulli measures corresponding to different values of the couple m, ϕ. In particular, there is a point in the phase diagram where there are three extremal equilibrium measures, corresponding to positive, zero and negative magnetization.

We study two Markov processes in the infinite volume spin configuration space Ω generated by self-adjoint operators in $L^2(\Omega, \mu)$, where μ is a Gibbs measure for some β, h_1, h_2 and finite γ: the so-called Glauber and Kawasaki dynamics. They can be described in words as follows: in the Glauber dynamics each spin at random times flips to a new value or stays unchanged with probabilities depending on the difference of energy before and after the flip. In the Kawasaki dynamics two neighbouring spins at random times exchange their values, or stay unchanged, with jump rates again depending on the energy difference. The latter stochastic evolution conserves the difference and the sum between the number of spins plus and minus (respectively, total magnetization and total concentration), while the former does not. Moreover, the jump rates depend on the magnetic fields h_1, h_2 in the Glauber dynamics and do not in the Kawasaki one. As a consequence, all the Gibbs measures regardless of the values of h_1, h_2 are invariant for the Kawasaki dynamics, while for the Glauber dynamics the only invariant measures are the Gibbs measures with the values of h_1, h_2 equal to those appearing in the jump rates.

We scale the lattice spacing by γ and look first at the behaviour of the system under the Glauber dynamics in the limit $\gamma \to 0$. We show that the empirical averages of magnetization and concentration converge weakly in probability to the solution of the set of two coupled non-local equations (3.7) and (3.8) (in Section 3).

To get a definite limit in the case of the Kawasaki dynamics we have to scale also the time as γ^{-2} (Giacomin et al., 1998). This is a process with two conservation laws. We prove also in this case a law of large numbers for the empirical averages of σ and σ^2, respectively, magnetization and concentration. Their limits satisfy the set of two coupled non-local second order integro-differential equations (3.5) in Section 3. These equations can be put in a nice form as a gradient flow of the free energy functional $\mathscr{F}$

$$\int \mathrm{d}r f^0(\underline{u}(r)) + \frac{1}{2}\int \mathrm{d}r \int \mathrm{d}r' J(r-r')[m(r)-m(r')]^2, \tag{1.2}$$

where $\underline{u} := (m,\phi)$ and $f^0(\underline{u})$ is

$$f^0(\underline{u}) := -m^2 + \phi + \beta^{-1}\left[\frac{1}{2}(m+\phi)\log(m+\phi) + \frac{1}{2}(\phi-m)\log(\phi-m) \right.$$
$$\left. +(1-\phi)\log(1-\phi) - \phi\log 2\right]. \tag{1.3}$$

Eqs. (3.5) become

$$\partial_t u_\alpha = \sum_{i=1}^{d}\sum_{\beta=1,2} \partial_i \left[M_{\alpha,\beta}\partial_i \frac{\delta\mathscr{F}}{\delta u_\beta}\right] \tag{1.4}$$

and in vectorial form

$$\partial_t \underline{u} = \nabla\cdot\left(M\nabla\frac{\delta\mathscr{F}}{\delta\underline{u}}\right), \tag{1.5}$$

where $\delta\mathscr{F}/\delta u_\alpha$ denotes the functional derivative of $\mathscr{F}$ with respect to u_α and M is the 2×2 mobility matrix

$$M = \beta(1-\phi)\begin{pmatrix} \phi + \dfrac{\phi^2-m^2}{1-\phi} & m \\ m & \phi \end{pmatrix}. \tag{1.6}$$

It is easy to see that $\mathscr{F}$ is a Lyapunov functional for (1.5). In fact,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{F} = -\sum_{i=1}^{d}\sum_{\alpha,\beta=1}^{2}\int \mathrm{d}r\, \partial_i\frac{\delta\mathscr{F}}{\delta u_\alpha} M_{\alpha,\beta}\partial_i\frac{\delta\mathscr{F}}{\delta u_\beta}. \tag{1.7}$$

The homogeneous minimizers of the functional $\mathscr{F}$ coincide with the minimizers of f^0, which has a unique minimizer but is not convex for β large enough. The convex envelope of f^0 is the free energy f of the KBC model at $\gamma=0$ and has some flat parts which single out a region F (*forbidden* region) in $D=\{(m,\phi)\colon \phi\in[0,1], m\leqslant\phi\}$, the domain of definition of f^0, such that for no value of the chemical potentials h_1, h_2 there is an extremal state with magnetization and concentration in F. Any Gibbs measure with averages m and ϕ in F has to be a linear superposition of the extremal states with $(m,\phi)\notin F$.

These properties of the energy functional should allow to relate concepts of stable, unstable and metastable phases with the behaviour of the solutions of (1.5).

Glauber and Kawasaki dynamics for the Ising model with Kac potential have been investigated thoroughly (Giacomin et al., 1998 and references therein), providing a microscopic description of segregation phenomena. We refer for Glauber to the series (De Masi et al., 1996a,b,c) and for Kawasaki to the papers by Giacomin and Lebowitz (1997, 1998), where the authors study the Kawasaki dynamics, with one conservation law, for the Ising model with Kac potential on a torus. Moreover, they prove the hydrodynamic limit by using Radon–Nicodym derivative methods and discuss the interface motion and the segregation behaviour (see for recent developments Carlen et al., 1999a,b).

We would notice that the mean field free energy functional for the Ising model is also not convex for β large and, since the first-order phase transition in the Ising model occurs at zero magnetic field, has a symmetric double-well structure. In the KBC model instead, the phase transition (in the sense of coexistence of phases) takes place at non-zero h_1 and/or h_2.

Finally, we have also studied a different kind of dynamics, which is in a way intermediate between Glauber and Kawasaki in the fact that it conserves only one quantity, the magnetization. Under this dynamics a bond (i,j) (namely a couple of neighbouring sites i and j) changes its configuration $(\sigma(i),\sigma(j))$ or stays unchanged, with probability depending on the energy difference, to a new configuration $(\sigma'(i),\sigma'(j))$ in such a way that in each site of the bond the spin variable changes by 1 and $\sigma(i)+\sigma(j)=\sigma'(i)+\sigma'(j)$. Hence the magnetization stays constant during the evolution, while the number of 0's can change. For example, changes from a bond configuration $(-1,1)$ to a configuration $(0,0)$ or from $(0,0)$ to $(-1,1)$ are possible: a sort of annihilation and creation process. The jump rates are chosen to satisfy the detailed balance with respect to the Gibbs measures for the Hamiltonian (1.1) with $h_2=1$. We derive under the diffusive scaling an equation for m, while ϕ on such a long time scale has already relaxed to the equilibrium and its effect can be seen in the mobility appearing in the equation for m. This dynamics is of the so-called non-gradient type (Spohn, 1991) and the proof of the hydrodynamic limit relies on the non-gradient method (Varadhan, 1994).

In Section 2 we describe the equilibrium properties of the KBC model and prove the limit $\gamma\to 0$ for the infinite volume free energy and pressure. In Section 3 we introduce the Glauber and Kawasaki dynamics and state the main theorems, whose proofs are contained in Sections 4 and 5. In Section 6 we prove the hydrodynamic limit for the non-gradient dynamics.

The proof of the hydrodynamic limit for the Kawasaki dynamics is based on the method of Guo et al. (1998). This method has been extended to infinite volume by Fritz (1990), by using a bound uniform in the volume for the entropy production. In the paper by Yau (1994) a different proof of the uniform entropy production bound has been given for Ginzburg–Landau models and in Landim and Mourragui (1997) this approach has been used to prove hydrodynamic limit for a class of zero range models. We follow the latter approach and prove a uniform bound for the entropy production, which is the time derivative of the entropy. Here we consider not the entropy but the relative entropy of the density of the process with respect to the Bernoulli measure v_{h_1,h_2} parametrized by the chemical potentials, which is not invariant for the process. Nevertheless, the bound of this production of entropy will be enough for the GPV method to work. In fact, it is easy to show that the Kawasaki dynamics (thought of as a lattice gas dynamics) is a weak perturbation (and reduces at $\beta=0$) of the following generalized symmetric exclusion process (GSEP): each particle on the lattice jumps at random times to a nearest-neighbour site x if and only if there is at most one particle in x. Hence, the state of the system on times γ^{-2} will be very close to the invariant measures for the GSEP process, which are the Bernoulli measures v_{h_1,h_2}. Therefore, the uniform bound for this entropy production will be sufficient to prove the hydrodynamic limit. The proof in the case of the Glauber dynamics is simpler: martingales methods are enough. In both cases it has been necessary to prove uniqueness theorems for the weak solutions of the limiting equations.

The non-gradient dynamics studied in Section 6 when formulated in the language of lattice gases is a weak perturbation of a non-gradient generalized simple exclusion process introduced in Kipnis et al. (1994). The diffusion coefficient for this process is not a constant, like in the symmetric exclusion process considered before, but a function of the density as a consequence of the non-gradient character of the dynamics. We work in this case in a torus, the extension to infinite volume being more involved because of the non-gradient nature of the problem. The proof is based again on the method of Guo et al. (1988) and on the non-gradient techniques of Varadhan (1994), that have to be adapted to deal with the perturbation. Also in this case, we will use as reference measure the Bernoulli measure, parametrized this time only by the magnetic field, which is not invariant for the dynamics. As a consequence, since the dynamics is non-gradient, in the limiting equation there is a new term related to the solution of the non-gradient problem for the unperturbed process. The presence of this term is crucial to recognize that the limiting equation is in the form of the gradient flux for a free energy functional. This is a general fact for non-gradient dynamics weakly perturbed by a Kac potential (see Giacomin et al., 2000). The limiting equation is

$$\partial_t m = \nabla \cdot \left(\Sigma \nabla \frac{\delta \mathscr{G}}{\delta m} \right)$$

with the energy functional $\mathscr{G}(m(r))$ of the form

$$\int \mathrm{d}r\, g^0(m(r)) + \frac{1}{2} \int \mathrm{d}r \int \mathrm{d}r' J(r-r')[m(r)-m(r')]^2, \tag{1.8}$$

where

$$\begin{aligned} g^0(m) := -m^2\beta^{-1}[&\tfrac{1}{2}(m+\phi(m))\log(m+\phi(m)) + \tfrac{1}{2}(\phi(m)-m)\log(\phi(m)-m) \\ &+ (1-\phi(m))\log(1-\phi(m)) - \phi(m)\log 2] \end{aligned} \tag{1.9}$$

and $\phi(m) = \langle \sigma^2 \rangle_{v_{h_1,0}}$ with h_1 determined as a function of m via $m(h_1) = \langle \sigma \rangle_{v_{h_1,0}}$. The mobility is given by the Einstein relation $\Sigma = D(m)\chi(m)$, with χ being the susceptivity and D the diffusion coefficient, which is given by the Green–Kubo formula (Kipnis et al., 1994). Note that $g^0(m)$ coincides with the functional f^0 in (1.3), associated to the Hamiltonian (1.1) for $h_2 = 1$, when evaluated in $(m, \phi(m))$. This is due to the fact that ϕ is a fast variable under this dynamics and in the diffusive limit it relaxes to its equilibrium value $\langle \sigma^2 \rangle_{v_{h_1,0}}$.

The convergence result that we get in this case is weaker than the one for Glauber and Kawasaki, because we are not able to prove the uniqueness of the hydrodynamic equation, due to the fact that the only regularity property known for the diffusion coefficient is continuity. Could the Lipschitz continuity for D be proven we would get a stronger convergence result.

2. The Kac–Blume–Capel model

The Blume–Capel model is a model of spins with values $0, \pm 1$ with nearest neighbours interactions, originally introduced to study the helium phase transition. Here we will introduce Kac–Blume–Capel (KBC) model which is a model of spins taking values in $\{-1,0,1\}$ on a d-dimensional lattice $\mathbb{Z}^d$ and interacting by means of a Kac potential.

A Kac potential is a function $J_\gamma(r)$, $\gamma > 0$, such that

$$J_\gamma(r) = \gamma^d J(\gamma r) \quad \text{for all } r \in \mathbb{R}^d,$$

where $J \in \mathscr{C}^2(\mathbb{R}^d)$ is a non-negative function supported in the unit ball, with $\int_{\mathbb{R}^d} J(r) = 1$ and $J(r) = J(-r)$ for all $r \in \mathbb{R}$.

The spin variable in the site $i \in \mathbb{Z}^d$ is denoted by $\sigma(i)$ and the infinite volume phase space by $\{-1,0,1\}^{\mathbb{Z}^d}$. A configuration is a function $\sigma : \mathbb{Z}^d \to \{-1,0,+1\}$, that is an element of $\Omega = \{-1,0,+1\}^{\mathbb{Z}^d}$. For any $\Lambda \subset \mathbb{Z}^d$, denote by σ_Λ the restriction to Λ of the configuration $\sigma, \sigma_\Lambda = \{\sigma(i),\ i \in \Lambda\}$.

The Gibbs measure, with potential $J_\gamma(r)$ and chemical potentials h_1, h_2 at inverse temperature $\beta > 0$, in a finite volume Λ and boundary condition ξ is the probability measure $\mu^{\beta,\xi}_{\gamma,\Lambda}$ on Ω:

$$\mu^{\beta,\xi}_{\gamma,\Lambda}(\sigma) = \frac{1}{Z^{\beta,\xi}_{\gamma,\Lambda}} \exp(-\beta H_\gamma(\sigma_\Lambda|\xi)),$$

where $Z^{\beta,\xi}_{\gamma,\Lambda}$ is the normalization constant and $H_\gamma(\sigma_\Lambda)$ is the formal Hamiltonian in a finite subset Λ of $\mathbb{Z}^d$, for the configuration σ_Λ

$$H_\gamma(\sigma_\Lambda) = \frac{1}{2} \sum_{\substack{i,j \in \Lambda \\ i \neq j}} J_\gamma(i-j)(\sigma(i) - \sigma(j))^2 - h_1 \sum_{i \in \Lambda} \sigma(i) - h_2 \sum_{i \in \Lambda} \sigma^2(i), \tag{2.1}$$

The infinite volume Gibbs measure $\mu_{\gamma,\beta}$ is a probability measure on Ω that can be constructed by some suitable limiting procedure.

The characteristics of models with Kac potentials is that the range of the interaction is γ^{-1} and the strength is γ^d, while the total interaction with all the other spins stays finite independently of γ. Hence, Kac potential interactions are useful to study the so-called mean field limit $\gamma \to 0$. The infinite volume free energy for the Kac Ising model has been computed in the limit $\gamma \to 0$ by Lebowitz and Penrose (1963), and the result agrees with (and gives rigorous support to) the van der Waals theory. The analogous result for the KBC model is

Theorem 2.1 (Lebowitz–Penrose limit). *Let $p_\gamma(\beta, h_1, h_2)$ be the pressure in the thermodynamic limit at $\gamma > 0$. Then*

$$\lim_{\gamma \to 0} p_\gamma(\beta, h_1, h_2) = \sup_{(m,\phi : |m| \leqslant \phi, \phi \leqslant 1)} [m^2 - \phi + \beta^{-1} s(m,\phi) + h_1 m + h_2 \phi], \tag{2.2}$$

where $s(m,\phi)$ is the entropy of a Bernoulli process in Ω with average spin equal to m and average square spin equal to ϕ (m is the magnetization and ϕ is called concentration).

$$s(m,\phi) = \tfrac{1}{2}(m+\phi)\log(m+\phi) - \tfrac{1}{2}(\phi-m)\log(\phi-m)$$
$$-(1-\phi)\log(1-\phi) + \phi\log 2. \tag{2.3}$$

The free energy $f(\beta,m,\phi)$ is defined as the Legendre transform of the pressure as

$$f(\beta,m,\phi) = \sup_{(h_1,h_2)} [h_1 m + h_2\phi - p(\beta,h_1,h-2)] = \mathrm{CE}[-m^2+\phi-\beta^{-1}s(m,\phi)], \tag{2.4}$$

where CE denotes the convex envelope. The complementary result in the canonical ensemble is

Theorem 2.2. *Define the free energy $f_\gamma(\beta,m,\phi)$ at $\gamma>0$ as follows: Consider the partition function in the canonical ensemble*

$$Z_\Lambda(N^-,N^0) = \sum_{\sigma\in\Gamma} \exp(-\beta H_\gamma(\sigma)), \tag{2.5}$$

where Γ is the set of configurations $\{\sigma\}\in\Omega_\Lambda$ such that the number of spins $\sigma=1$ is fixed to be N^- and the number of spins $\sigma=0$ is N^0. Let N be the total number of spins in a finite volume Λ and put

$$m = \frac{1}{N}\sum_{i=1}^{N}\sigma(i) = \frac{1}{N}[N-2N^-N^0],$$
$$\phi = \frac{1}{N}\sum_{i=1}^{N}(\sigma(i))^2 = \frac{1}{N}[N-N^0]. \tag{2.6}$$

The free energy at $\gamma>0$ in the thermodynamic limit is defined as

$$\beta f_\gamma(\beta,m,\phi) := \lim_{\Lambda,N^-,N^0\to\infty} N^{-1}\log Z_\Lambda(N^-,N^0),$$

where the limit is taken in such a way that (2.6) *holds.*

Then

$$\lim_{\gamma\to 0} f_\gamma(\beta,m,\phi) = \mathrm{CE}[-m^2+\phi-\beta^{-1}s(m,\phi)]. \tag{2.7}$$

The proof of this theorem is similar to the one of the Lebowitz–Penrose theorem (Lebowitz and Penrose, 1963), (see also De Masi and Presutti (1991)) and will not be given explicitly here. We only remark that the proof of Lebowitz and Penrose (1963) is based on a block spin renormalization procedure and the main point in the proof is writing the renormalized Hamiltonian for the block spins (whose expression will depend on the form of the interaction). Since the interaction term in the KBC model is a two-body interaction like in the Ising model, this part of the proof goes through in almost the same way. Obviously, the entropy will depend on the values of the spin and is in fact different from the one computed in Lebowitz and Penrose (1963).

The phase structure of the model at $\gamma=0$ is very rich. To discuss the phase transition we can for example examine the function $p^0 := m^2 - \phi + \beta^{-1}s(m,\phi) + h_1 m + h_2\phi$

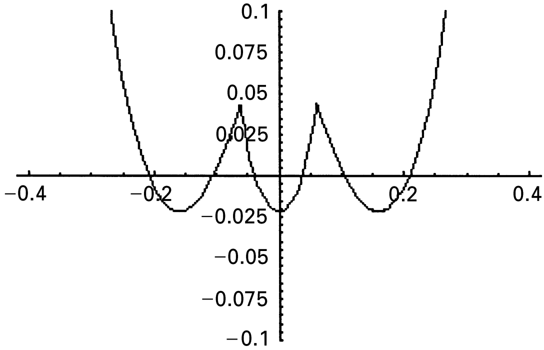

Fig. 1. $-p^0(m,\phi(m))$ at $\beta=3.76,\ h_1=0,\ h_2=0.06$.

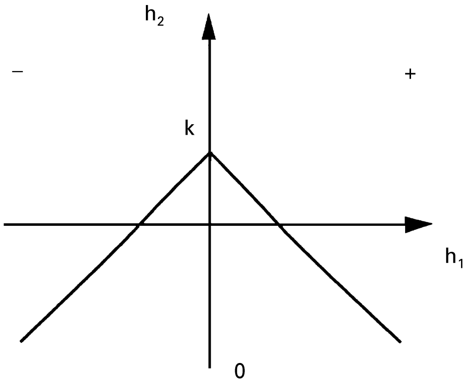

Fig. 2. Phase diagram at $\beta \gg \beta_c$.

determining the pressure in (2.2). The extremals of the function p^0 are determined by the equations

$$m=\phi\tanh(2\beta m+\beta h_1),$$

$$\phi=\exp\{\beta(h_2-1)\}(1-\phi)\cosh(2\beta m+\beta h_1). \tag{2.8}$$

These equations can be solved numerically. For $\beta \leqslant \bar{\beta}_c = \frac{1}{2}$ there is only one solution, while for $\beta > \bar{\beta}_c$ the equations admit more than one solution and the function can have more than one maximum for suitable values of h_1 and h_2. For β in the interval $[\frac{1}{2},\frac{3}{2}]$ there is in the plane β, h_2 a line of second-order phase transition, which changes to first order (made of triple points) at $\beta=\bar{\beta}_c$ and $h_2=1+\frac{4}{3}\ln 2$. The point $\beta=\frac{3}{2},\ h_1=0$, $h_2=1+\frac{4}{3}\ln 2$ is called tricritical point. We refer for details to the paper (Blume et al., 1971). In Fig. 1 there is the graph of $-p^0$ as a function of m (by means of (2.8)) at a three-phase coexistence point.

The phase diagram in the plane h_1, h_2, for β large is shown in Fig. 2. There are three lines of phase coexistence stemming from a triple point which separate the one-phase regions. In the semiplane $h_1>0$ ($h_1<0$) there is a line of coexistence of phases with

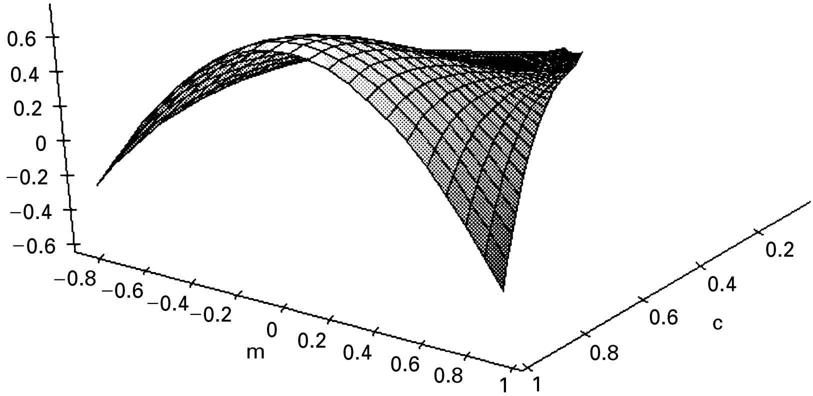

Fig. 3. $f^0(m,\phi)$ at $\beta=2$.

positive (negative) and zero magnetization and along the line $h_1=0$, $h_2>k$ there is coexistence of phases with positive and negative magnetization.

Finally, we note that the function

$$f^0 := -m^2+\phi-\beta^{-1}s(m,\phi)$$

is not convex for $\beta>\frac{1}{2}$. In fact

$$\mathrm{Hess}(f^0)=\frac{2\beta(m^2-\phi)+1}{(\beta^2(1-\phi)(\phi^2-m^2))}. \tag{2.9}$$

The Hessian is negative in the region $\phi-m^2>(2\beta)^{-1}$. Hence f, the free energy at $\gamma=0$ defined in (2.4) as the convex envelope of f^0, has some flat parts for $\beta>\frac{1}{2}$, corresponding to regions in $D=\{m,\phi\colon\ \phi\in[0,1], m\leqslant\phi\}$ such that the vectorial function $\underline{h}(m,\phi)$, $\underline{h}:=(h_1,h_2)$ cannot be inverted. We call this region in D as a forbidden region and denote it by F. A point $(\bar{m},\bar{\phi})\in F$ has the property that the equations $m(h_1,h_2)=\bar{m}, \phi(h_1,h_2)=\bar{\phi}$ cannot be solved for (h_1,h_2) (Fig. 3).

Remark. This model can also be looked at as a lattice gas of two species of particles such that in each site of the lattice there is at most one particle for each species. A way of realizing the correspondence is for example the following. Call $\eta_{\rm b}(i)=1,0$, $\eta_{\rm r}(i)=1,0$ the occupation number in the site i of the particles of colours blue and red, respectively. Then, the relation $\sigma(i)=\eta_{\rm b}(i)-\eta_{\rm r}(i)$ determines a lattice gas of blue and red particles with repulsive interaction between particles of the same colour and attractive interaction between particles of different colours. Under this correspondence a configuration of particles η with two particles in a site i is identical to a configuration ζ with no particles in i.

Finally, we note that the relation $\sigma(i)=\eta(i)-1$ links the model to a lattice gas with one species of particles with at most two particles per site.

3. Glauber and Kawasaki dynamics

We consider two kinds of dynamics for the spin system introduced in the previous section: the Glauber and Kawasaki dynamics, the latter conserving both magnetization and concentration.

For $(i,j) \in \mathbb{Z}^d \times \mathbb{Z}^d$, $k \in \mathbb{Z}^d$, $\sigma \in \Omega$ and any cylinder function $F : \Omega \to \mathbb{R}$, define $(\nabla_{i,j}F)(\sigma)$, $(\nabla_k^+ F)(\sigma)$ and $(\nabla_k^- F)(\sigma)$ by

$$(\nabla_{i,j}F)(\sigma) = F(\sigma^{i,j}) - F(\sigma),$$

$$(\nabla_k^\pm F)(\sigma) = F(\sigma^{\pm,k}) - F(\sigma),$$

where $\sigma^{i,j}$ is the configuration obtained from σ by interchanging the values at i and j:

$$(\sigma^{i,j})(l) = \begin{cases} \sigma(l) & \text{if } l \neq i,j, \\ \sigma(j) & \text{if } l = i, \\ \sigma(i) & \text{if } l = j, \end{cases}$$

and $\sigma^{\pm,k}$ is defined as

$$(\sigma^{\pm,k})(l) = \begin{cases} \sigma(l) & \text{if } l \neq k, \\ \sigma(k) \pm 1 \bmod 3 & \text{if } l = k. \end{cases}$$

The Kawasaki dynamics with parameter $\beta \geqslant 0$ is the unique Markov process on Ω, whose pregenerator $\mathbb{L}_\gamma^{K,\beta}$ acts on the cylinder functions as

$$(\mathbb{L}_\gamma^{K,\beta} f)(\sigma) = \sum_{\substack{i,j \in \mathbb{Z}^d \\ |i-j|=1}} C_\gamma^{K,\beta}(i,j;\sigma)[(\nabla_{i,j} f)(\sigma)].$$

Here and in the following $|.|$ stands for the max norm of $\mathbb{R}^d$. For $(i,j) \in \mathbb{Z}^d \times \mathbb{Z}^d$ and $\sigma \in \Omega$, the rate $C_\gamma^{K,\beta}(i,j;\sigma)$ is given by

$$C_\gamma^{K,\beta}(i,j;\sigma) = \Phi\{\beta(\nabla_{i,j} H_\gamma(\sigma))\}.$$

Here $\Phi : \mathbb{R} \to \mathbb{R}_+$ is a continuously differentiable function in a neighbourhood of 0, such that $\Phi(0) = 1$ and satisfies the detailed balance condition (cf. Giacomin and Lebowitz, 1997; Giacomin et al., 1998)

$$\Phi(E) = \exp(-E)\Phi(-E). \tag{3.1}$$

The generator of the Glauber evolution is given by

$$(\mathbb{L}_\gamma^{G,\beta} f)(\sigma) = \sum_{i \in \mathbb{Z}^d} C_\gamma^{G,\pm}(i;\sigma)[(\nabla_i^\pm f)(\sigma)],$$

where the rates $C_\gamma^{G,\pm}(i;\sigma)$ are defined as

$$C_\gamma^{G,\pm}(i;\sigma) = \frac{1}{2} \frac{1}{1 + \exp(\beta(\nabla_i^\pm H_\gamma))}$$

corresponding to the choice $\Phi(E) = \frac{1}{2}[1 + \exp E]^{-1}$.

Notice that the quantities $(\nabla_{i,j} H_\gamma)$ and $(\nabla_i^\pm H_\gamma)$ are well defined since they involve only a finite number of non-zero differences. For the proof of the existence and uniqueness of these Markov processes, we refer to Liggett (1985).

In the case of Kawasaki dynamics, if $\beta = 0$, the evolution reduces to a simple known process. In the setting of the lattice gas with one species of particles this dynamics is a generalized simple exclusion process GSEP (Kipnis and Landim, 1999) with rate one, and we shall denote its pregenerator simply by $\mathbb{L}^0$. It differs from the usual SEP for the exclusion rule involved: in each point are allowed at most two particles. We shall see in Section 3 that the dynamics with $\beta > 0$ is a weak perturbation of this simple exclusion. As explained in the remark in Section 3, the Kawasaki dynamics can also be interpreted as the motion of two species of particles, moving as a symmetric simple exclusion process with rate one, with the exclusion rule $\eta_{\mathrm{b}} + \eta_{\mathrm{r}} \leqslant 1$, such that also jumps exchanging colours between neighbour sites are allowed. If such jumps are forbidden the system becomes a non-gradient system and the diffusion coefficient in this case is different from one (Quastel, 1992).

Since the Kawasaki dynamics conserves magnetization and concentration the invariant measures will be Gibbs measures parametrized by two chemical potentials. It is useful to introduce the invariant measures for the exclusion process GSEP, which are Bernoulli measures depending on two parameters. For each positive integer n, denote by $\Lambda_n \subset \mathbb{Z}^d$ the sublattice of size $(2n+1)^d, \Lambda_n = \{-n,\dots,n\}^d$. For $A = (a,b) \in [-1,1] \times [0,1]$, we define $\bar{\nu}_A$ as the product measure on Ω with chemical potential A such that, for all positive integers n, the restriction $\bar{\nu}_{A,n}$ of $\bar{\nu}_A$ to Ω_n is given by

$$\mathrm{d}\bar{\nu}_{A,n} = Z_{A,n}^{-1} \exp\left\{ a \sum_{i\in\Lambda_n} \sigma(i) + b \sum_{i\in\Lambda_n} \sigma^2(i) \right\},$$

where $Z_{A,n}$ is the normalization constant. For $(a,b) \in [-1,1] \times [0,1]$ let $m = m(a,b)$ (resp. $\phi = \phi(a,b)$) be the expectation of $\sigma(0)$ (resp. $\sigma^2(0)$) under $\bar{\nu}_{A,n}$:

$$m(a,b) = \boldsymbol{E}^{\bar{\nu}_{A,n}}(\sigma(0)),$$

$$\phi(a,b) = \boldsymbol{E}^{\bar{\nu}_{A,n}}(\sigma^2(0)).$$

Observe that the function Ψ defined on $]-1,1[\times]0,1[$ by $\Psi(a,b) = (m,\phi)$ is a bijection from $]-1,1[\times]0,1[$ to $I = \{(m,\phi)\colon 0 < \phi < 1, -1 < m < \phi\}$. For every $P = (m,\phi) \in I$, we denote by $\nu_{P,n}$ the product measure such that

$$m = \boldsymbol{E}^{\nu_{P,n}}[\sigma(0)],$$

$$\phi = \boldsymbol{E}^{\nu_{P,n}}[\sigma^2(0)]. \tag{3.2}$$

We take γ^{-1}, the range of the interaction, as macroscopic space unit and consider the limit $\gamma \to 0$. We want to establish for both Kawasaki and Glauber dynamics a law of large numbers for the empirical fields corresponding to magnetization and concentration.

In the Glauber case we look at the behaviour of the fields for finite time, while in the Kawasaki case the relevant time scale is γ^{-2}. Fix a sequence of probability measures $(\mu^\gamma)_\gamma$, associated to the same initial profile $(m_0,\phi_0)\colon \mathbb{R}^d \times \mathbb{R}^d \to I$ in the following sense:

$$\lim_{\gamma\to 0}\mu^\gamma\left\{\left[\left|\gamma^d\sum_{i\in\mathbb{Z}^d}U(i\gamma)\sigma(i)-\int_{\mathbb{R}^d}U(x)m_0(x)\,\mathrm{d}x\right|\right.\right.$$
$$\left.\left.+\left|\gamma^d\sum_{i\in\mathbb{Z}^d}V(i\gamma)\sigma(i)^2-\int_{\mathbb{R}^d}V(x)\phi_0(x)\,\mathrm{d}x\right|\right]>\delta\right\}=0 \tag{3.3}$$

for every continuous function $U,V:\mathbb{R}^d\to\mathbb{R}$ with compact support, and every $\delta>0$.

Denote by $\boldsymbol{P}_{\mu^\gamma}^{(\mathrm{K}),\beta}$ (resp. $\boldsymbol{P}_{\mu^\gamma}^{(\mathrm{G}),\beta}$) the probability measure on the path space $D(\mathbb{R}_+,\Omega)$ corresponding to the Markov process $(\sigma(t,.))_{t\geqslant 0}$ with the generator $\gamma^{-2}\mathbb{L}_\gamma^{K,\beta}$ (resp. $\mathbb{L}_\gamma^{G,\beta}$), and starting from μ^γ, and by $\boldsymbol{E}_{\mu^\gamma}^{(\mathrm{K}),\beta}$ (resp. $\boldsymbol{E}_{\mu^\gamma}^{(\mathrm{G}),\beta}$) the expectation with respect to $\boldsymbol{P}_{\mu^\gamma}^{(\mathrm{K}),\beta}$ (resp. $\boldsymbol{P}_{\mu^\gamma}^{(\mathrm{G}),\beta}$). Denote by $\mathscr{C}_K^2(\mathbb{R}^d)$ the space of real twice continuously differentiable functions with compact support.

The main result of this section is the following theorem.

Theorem 3.1. *Under* (3.3), *for any* $\delta>0$, $t\geqslant 0$ *and* $U,V\in\mathscr{C}_K^2(\mathbb{R}^d)$ *the following holds*:

(i) *Kawasaki dynamics*:

$$\lim_{\gamma\to 0}\boldsymbol{P}_{\mu^\gamma}^{(\mathrm{K}),\beta}\left\{\left[\left|\gamma^d\sum_{i\in\mathbb{Z}^d}U(i\gamma)\sigma(t,i)-\int_{\mathbb{R}^d}U(x)m(t,x)\,\mathrm{d}x\right|\right.\right.$$
$$\left.\left.+\left|\gamma^d\sum_{i\in\mathbb{Z}^d}V(i\gamma)\sigma(t,i)^2-\int_{\mathbb{R}^d}V(x)\phi(t,x)\,\mathrm{d}x\right|\right]>\delta\right\}=0, \tag{3.4}$$

where (m,ϕ) *is the unique weak solution of*

$$\begin{aligned}
&\partial_t m=\nabla\cdot[\nabla m-2\beta(\phi-(m)^2)(\nabla J*m)],\\
&\partial_t\phi=\nabla\cdot[\nabla\phi-2\beta m(1-\phi)(\nabla J*m)],\\
&m(0,\cdot)=m_0,\quad \phi(0,\cdot)=\phi_0
\end{aligned}\tag{3.5}$$

where $*$ *denotes the convolution on the spatial variable.*

(ii) *Glauber dynamics*:

$$\lim_{\gamma\to 0}\boldsymbol{P}_{\mu^\gamma}^{(\mathrm{G}),\beta}\left\{\left[\left|\gamma^d\sum_{i\in\mathbb{Z}^d}U(i\gamma)\sigma(t,i)-\int_{\mathbb{R}^d}U(x)m^{(\mathrm{G})}(t,x)\,\mathrm{d}x\right|\right.\right.$$
$$\left.\left.+\left|\gamma^d\sum_{i\in\mathbb{Z}^d}V(i\gamma)\sigma(t,i)^2-\int_{\mathbb{R}^d}V(x)\phi^{(\mathrm{G})}(t,x)\,\mathrm{d}x\right|\right]>\delta\right\}=0, \tag{3.6}$$

where $(m^{(\mathrm{G})}, \phi^{(\mathrm{G})})$ *is the unique weak solution of*

$$\begin{cases} \partial_t \begin{pmatrix} m^{(\mathrm{G})} \\ \phi^{(\mathrm{G})} \end{pmatrix} = \mathscr{G}^\beta \begin{pmatrix} m^{(\mathrm{G})} \\ \phi^{(\mathrm{G})} \end{pmatrix} := \begin{pmatrix} \mathscr{G}_1^\beta(m^{(\mathrm{G})}, \phi^{(\mathrm{G})}) \\ \mathscr{G}_2^\beta(m^{(\mathrm{G})}, \phi^{(\mathrm{G})}) \end{pmatrix} \\ \begin{pmatrix} m^{(\mathrm{G})}(0,.) \\ \phi^{(\mathrm{G})}(0,.) \end{pmatrix} = \begin{pmatrix} m_0 \\ \phi_0 \end{pmatrix} \end{cases} \tag{3.7}$$

and $\mathscr{G}_1^\beta, \mathscr{G}_2^\beta$ *are defined as*

$$\begin{aligned} \mathscr{G}_1^\beta(m,\phi) = {} & \frac{1}{4}\left\{\tanh\frac{\beta}{2}(\alpha+h') + \tanh\frac{\beta}{2}(\alpha-h')\right\} \\ & + \frac{1}{4}\frac{m}{2}\left\{\tanh\frac{\beta}{2}(\alpha+h') - \tanh\frac{\beta}{2}(\alpha-h')\right\} - \frac{3}{4}m \\ & + \frac{\phi}{4}\left\{2\tanh\beta\alpha - \frac{1}{2}\tanh\frac{\beta}{2}(\alpha+h') - \frac{1}{2}\tanh\frac{\beta}{2}(\alpha-h')\right\}, \\ \mathscr{G}_2^\beta(m,\phi) = {} & \frac{1}{4}\left\{\tanh\frac{\beta}{2}(\alpha+h') - \tanh\frac{\beta}{2}(\alpha-h')\right\} \\ & + \frac{1}{4}\frac{m}{2}\left\{\tanh\frac{\beta}{2}(\alpha-h') + \tanh\frac{\beta}{2}(\alpha+h')\right\} \\ & - \frac{1}{4}\frac{\phi}{2}\left\{\tanh\frac{\beta}{2}(\alpha+h') - \tanh\frac{\beta}{2}(\alpha-h')\right\} + \frac{1}{4}(2-3\phi). \end{aligned} \tag{3.8}$$

Here

$$\alpha = 2m * J + h_1, \quad h' = h_2 - \int_{\mathbb{R}^d} J(x)\,\mathrm{d}x = h_2 - 1$$

and $*$ *denotes the convolution.*

The limiting equations for the Kawasaki dynamics can be rewritten in a nice form as a gradient flux associated with the local mean field free energy functional. Put $\underline{u} := (m,\phi)$. Define the free energy functional as

$$\mathscr{F}(\underline{u}) := -\frac{1}{\beta}\int \mathrm{d}r\, s(\underline{u}(r)) - \int \mathrm{d}r \int \mathrm{d}r'\, J(r-r')m(r)m(r')$$

and the mobility matrix as

$$M = -\beta[\mathrm{Hess}(s)]^{-1} = \beta(1-\phi)\begin{pmatrix} \phi + \dfrac{\phi^2 - m^2}{1-\phi} & m \\ m & \phi \end{pmatrix}.$$

Then Eqs. (3.5) become

$$\partial_t \underline{u} = \nabla\left(M\nabla\frac{\delta\mathscr{F}}{\delta\underline{u}}\right).$$

Writing the free energy functional $\mathscr{F}$ in the equivalent form

$$\int \mathrm{d}r\, f^0(\underline{u}(r)) + \frac{1}{2}\int \mathrm{d}r \int \mathrm{d}r'\, J(r-r')[m(r)-m(r')]^2$$

with

$$f^0(\underline{u}) := -m^2 + \phi - \beta^{-1} s(\underline{u}),$$

we see that $\mathscr{F}$ reduces for homogeneous profiles of magnetization and concentration to the non-convex free energy f^0 of the KBC model, so that the stationary homogeneous solutions of Eqs. (3.5) coincide with the solutions of (2.8). Moreover, $\mathscr{F}$ is a Lyapunov functional for the evolution, namely it is decreasing in time along the solutions of Eqs. (3.5). This follows from (1.7) and the positivity of the matrix M.

On the contrary, in the Glauber case the limiting equations (3.7)–(3.8) are rather messy. It is not even known if the energy functional is a Lyapunov functional: we have only numerical evidence.

The region D in the plane (m,ϕ), such that $D = \{m,\phi\colon 0 \leqslant \phi \leqslant 1, |m| \leqslant \phi,\}$ can be partitioned for any fixed $\beta \geqslant \bar{\beta}_c$ in three parts:

(a) the unstable region $U = \{(m,\phi) \in F\colon \phi - m^2 \geqslant (2\beta)^{-1}\}$, where F is the forbidden region defined after (2.9),
(b) the metastable region $M = \{(m,\phi) \in F\colon \phi - m^2 < (2\beta)^{-1}\}$,
(c) the stable region $D - (U \cup M)$.

The segregation phenomena may appear by choosing an initial datum corresponding to total magnetization and concentration in the unstable region. One expects that a stationary solution of Eqs. (3.5) with this initial condition be unstable.

4. Dirichlet form estimates for Kawasaki dynamics

The proof of Theorem 3.1 is based on a priori estimates uniform in the volume for the entropy and the Dirichlet form, which are given in this section. For each positive integer n and a measure μ on $\Omega_n = \{-1,0,1\}^{\Lambda_n}$, we denote by μ_n the marginal of μ on Ω_n,

$$\mu_n(\xi) = \mu\{\sigma\colon \sigma(i) = \xi(i) \text{ for } |i| \leqslant n\} \quad \text{for each } \xi \in \Omega_n.$$

For a chemical potential P, and a positive integer n, we denote by $s_n(\mu_n | v_{P,n})$ the relative entropy of μ_n with respect to $v_{P,n}$

$$s_n(\mu_n | v_{P,n}) = \sup_{U \in C_b(\Omega_n)} \left\{ \int U(\sigma)\,d\mu_n(\sigma) - \log \int e^{U(\sigma)}\,dv_{P,n}(\sigma) \right\}.$$

In this formula $C_b(\Omega_n)$ stands for the space of all functions on Ω_n. Since the measure $v_{P,n}$ gives a positive probability to each configuration, all the measures on Ω_n are absolutely continuous with respect to $v_{P,n}$ and we have an explicit formula for the entropy:

$$s_n(\mu_n | v_{P,n}) = \int \log(f_n(\sigma))\,d\mu_n(\sigma),$$

where f_n is the probability density of μ_n with respect to $\nu_{P,n}$. Notice that by the entropy convexity and since $\sup_\sigma \sup_i |\sigma(i)|$ is finite, we have

$$s_n(\mu_n|\nu_{P,n}) \leqslant C_0 n^d \tag{4.1}$$

for some constant C_0 that depends on P (cf. Kipnis and Landim, 1999).

Define the Dirichlet form $D_n(\mu_n|\nu_{P,n})$ of the measure μ_n with respect to $\nu_{P,n}$ associated to the exclusion process by

$$\begin{aligned} D_n(\mu_n|\nu_{P,n}) &= -\int \sqrt{f_n}(\sigma)(\mathbb{L}_n^0\sqrt{f_n})(\sigma)\,\mathrm{d}\nu_{P,n}(\sigma) \\ &= \sum_{\substack{i,j\in\Lambda_n \\ |i-j|=1}} I_{i,j}(f_n), \end{aligned}$$

where $I_{i,j}(.)$ is given by

$$I_{i,j}(f_n) = -\int \sqrt{f_n}(\sigma)(\mathbb{L}_{i,j}^0\sqrt{f_n})(\sigma)\,\mathrm{d}\nu_{P,n}(\sigma)$$

and $\mathbb{L}_n^0$ is the restriction of the process to the box Λ_n

$$\mathbb{L}_n^0 = \sum_{\substack{i,j\in\Lambda_n \\ |i-j|=1}} \mathbb{L}_{i,j}^0.$$

Here for a bond $(i,j) \in \mathbb{Z}^d \times \mathbb{Z}^d$, $\mathbb{L}_{i,j}^0$ stands for the piece of generator associated to the exchange of spins between sites i and j for the exclusion process.

Define the entropy $\mathscr{S}(\mu|\nu_P)$ and the Dirichlet form $\mathscr{D}(\mu|\nu_P)$ of a measure μ on Ω with respect to ν_P as

$$\mathscr{S}(\mu|\nu_P) = \gamma\sum_{n\geqslant 1} s_n(\mu_n|\nu_{P,n})\mathrm{e}^{-\gamma n},$$

$$\mathscr{D}(\mu|\nu_P) = \gamma\sum_{n\geqslant 1} D_n(\mu_n|\nu_{P,n})\mathrm{e}^{-\gamma n}.$$

Notice that by (4.1), there exists a positive constant C depending on P such that for any probability measure μ on Ω

$$\mathscr{S}(\mu|\nu_P) \leqslant C\gamma^{-d}. \tag{4.2}$$

Through this section we consider Kawasaki dynamics with fixed parameter $\beta > 0$ and with fixed scaling parameter γ^{-1}. We shall denote by $(S_\gamma^{\mathrm{K},\beta}(t))_{t\geqslant 0}$ the semigroup associated to the generator $\gamma^{-2}\mathbb{L}_\gamma^{\mathrm{K},\beta}$ (that is, the semigroup of Kawasaki dynamics with parameter β, accelerated by γ^{-2}). For a measure μ on Ω we shall denote by $\mu^{\mathrm{K},\beta}(t)$ the time evolution of the measure μ under the semigroup $S_\gamma^{\mathrm{K},\beta}: \mu^{\mathrm{K},\beta}(t) = \mu S_\gamma^{\mathrm{K},\beta}(t)$.

When $\beta=0$, the process reduces to the generalized simple exclusion process (Lemma 4.1), and in particular the product measures are invariant for the generator $\mathbb{L}_\gamma^{\mathrm{K},0}$. In this case, by using the methods of Fritz (1990) and Yau (1994) one can get entropy and the Dirichlet form estimates uniform in the volume, for the entropy of processes evolving in large finite volumes and then extend them by lower semi-continuity to the infinite system. For $\beta \neq 0$ product measures are no more invariant for the generator, but it is possible to take advantage of the fact that the process is a weak perturbation of the exclusion one (Lemma 4.1 below) and adapt Fritz's approach to that case without considering an approximation of the infinite volume dynamics. Notice that from (4.2) there is no need for an initial condition on the entropy in Theorem 3.1.

Lemma 4.1. *For any $i \in \mathbb{Z}^d$, unit vector $e \in \mathbb{Z}^d$ and $\sigma \in \Omega$*

$$\begin{aligned} C_\gamma^{\mathrm{K},\beta}(i,i+e;\sigma) &= 1 - \gamma\beta[\sigma(i+e)-\sigma(i)]\gamma^d \sum_{\ell\in\mathbb{Z}^d} (e\cdot\nabla J)(\gamma(i-\ell))\sigma(\ell) \\ &\quad -\gamma^{d+1}\beta[\sigma(i+e)-\sigma(i)]^2(e\cdot\nabla J)(0) + \mathrm{O}(\gamma^2) \\ &= 1+\beta\mathrm{O}(\gamma). \end{aligned}$$

Proof. By definition of H_γ, for all $i,j \in \mathbb{Z}^d$ and $\sigma \in \Omega$

$$\begin{aligned} (\nabla_{i,j}H_\gamma)(\sigma) &= 2[\sigma(i)-\sigma(j)]\gamma^d \sum_{\ell\in\mathbb{Z}^d} [J_\gamma(i-\ell)-J_\gamma(j-\ell)]\sigma(\ell) \\ &\quad +2[\sigma(i)-\sigma(j)]^2\gamma^d[J_\gamma(i-j)-J_\gamma(0)]. \end{aligned} \tag{4.3}$$

To prove the lemma, it is enough to remark that the conditions imposed on Φ imply that $\Phi'(0) = -\frac{1}{2}$ (cf. Giacomin and Lebowitz, 1997) and to use Taylor expansion. □

We get the following estimate for the Dirichlet form in the infinite volume:

Theorem 4.2. *There exists a positive finite constant C_1 that depends on P, t and β such that*

$$\int_0^t \mathscr{D}(\mu^{\mathrm{K},\beta}(\tau)|\nu_P)\,\mathrm{d}\tau \leqslant C_1\gamma^{2-d}.$$

The strategy of the proof is to introduce a suitable entropy and Dirichlet form in finite volume and bound the corresponding entropy production in terms of the finite volume Dirichlet form times γ^2 uniformly in the volume (Lemma 4.2). Then, the a priori bound on the entropy (4.2) allows to get the estimate.

Fix a measure μ on Ω and a chemical potential P. For every $t \geqslant 0$ and positive integer n, denote by f_n^t the probability density of $(\mu^{\mathrm{K},\beta}(t))_n$ with respect to $\nu_{P,n}$. To simplify the notation, we denote respectively by $s_n(f_n^t)$ and $D_n(f_n^t)$ the entropy and the Dirichlet form of $(\mu^{\mathrm{K},\beta}(t))_n$ with respect to $\nu_{P,n}$. For all positive integers M, let M_γ be defined by $M_\gamma = N_\gamma^2 + M$, where $N_\gamma = [\![\gamma^{-1}]\!]$ stands for the integer part of γ^{-1}. Define respectively the entropy $\mathscr{S}_{M_\gamma}(.)$ and the Dirichlet form $\mathscr{D}_{M_\gamma}(.)$ with finite sum by

$$\mathscr{S}_{M_\gamma}(\mu^{\mathrm{K},\beta}(t)|\nu_P) = \gamma \sum_{n=1}^{M_\gamma} s_n(f_n^t)\mathrm{e}^{-n\gamma},$$

$$\mathscr{D}_{M_\gamma}(\mu^{\mathrm{K},\beta}(t)|\nu_P) = \gamma \sum_{n=1}^{M_\gamma} D_n(f_n^t)\mathrm{e}^{-n\gamma}.$$

Lemma 4.3. *There exist positive and finite constants A_0 and A_1 that depend on P and β such that, for all positive M*

$$\partial_t \mathscr{S}_{M_\gamma}(\mu^{\mathrm{K},\beta}(t)|\nu_P) \leqslant -\gamma^{-2} A_0 \mathscr{D}_{M_\gamma}(\mu^{\mathrm{K},\beta}(t)|\nu_P) + A_1 \gamma^{-d}. \tag{4.4}$$

Before proving the lemma we conclude the proof of Theorem 4.2.

Proof of Theorem 4.2. Integrate (4.4) from 0 to t, let $M \uparrow \infty$ and use (4.2). □

Proof of Lemma 4.3. We drop the indices in $\mathbb{L}_\gamma^{\mathrm{K},\beta}$ and denote the generator simply by $\mathbb{L}$. For all positive integers k denote by $\mathbb{L}_k$ the restriction of the generator $\mathbb{L}$ to the box Λ_k. For a subset $A \subset \Lambda$ of $\mathbb{Z}^d$, and a function h in $L^1(\nu_{P,\Lambda})$, let $\langle h \rangle_A$ be the function on $\{-1,0,1\}^{\Lambda\setminus A}$ obtained by integrating h over the coordinates $\{\sigma(x)\colon x \in A\}$ with respect to $\nu_{P,\Lambda}$. When $A = \Lambda_{n+m+1} - \Lambda_n$, we shall denote this expectation simply by $\langle h \rangle_{\Lambda_n^m}$.

With this notation, we can verify that f_n^t satisfies the equation

$$\partial_t f_n^t = \gamma^{-2} \langle \mathbb{L}_{n+1}^* f_{n+N_\gamma+1}^t \rangle_{\Lambda_n^{n+N_\gamma}}, \tag{4.5}$$

where for a positive integer k, $\mathbb{L}_k^*$ represents the adjoint operator of $\mathbb{L}_k$ in $L^2(\nu_{P,\Lambda_k})$. By relation (4.5) and the explicit formula for the entropy we have that

$$\begin{aligned} \partial_t s_n(f_n^t) &= \gamma^{-2} \int f_{n+N_\gamma+1}^t \mathbb{L}_n \log(f_n^t)\, \mathrm{d}\nu_{P,n+N_\gamma+1} \\ &\quad + \gamma^{-2} \int f_{n+N_\gamma+1}^t (\partial L_{n+1}) \log(f_n^t)\, \mathrm{d}\nu_{P,n+N_\gamma+1} \\ &:= \Omega_n^1 + \Omega_n^2. \end{aligned} \tag{4.6}$$

The first term Ω_n^1 on the right-hand side of the last inequality corresponds to the exchanges in the interior of Λ_n, while the second term Ω_n^2 is associated to exchanges at the boundary

$$(\partial L_{n+1})(f) = \sum_{\substack{i \in \Lambda_n, j \notin \Lambda_n \\ |i-j|=1}} C_\gamma^{\mathrm{K},\beta}(i,j;\sigma)[(\nabla_{i,j} f)(\sigma)].$$

The proof is divided into three steps. In the first two steps we estimate Ω_n^1 and Ω_n^2 and in the third one we prove (4.4).

Step 1 (*bound of* Ω_n^1). Fix a bond $(i,j) \in \Lambda_n \times \Lambda_n$ such that $|i-j| = 1$, denote by $\mathbb{L}_{i,j}$ the one bond generator corresponding to the exchange of spins between i and j and let $F_n^{i,j}(\sigma)$ be the function defined by $F_n^{i,j}(\sigma) = \langle (C_\gamma^{\mathrm{K},\beta}(i,j;\cdot)/f_n^t(\sigma)) f_{n+N_\gamma+1}^t(\cdot) \rangle_{\Lambda_n^{n+N_\gamma}}$. We have

$$\gamma^{-2} \int f_{n+N_\gamma+1}^t \mathbb{L}_{i,j} \log(f_n^t)\, \mathrm{d}\nu_P = \gamma^{-2} \int F_n^{i,j}(\sigma) f_n^t(\sigma) \log\left\{ \frac{f_n^t(\sigma^{i,j})}{f_n^t(\sigma)} \right\} \mathrm{d}\nu_P(\sigma).$$

Using the basic inequality

$$a(\log b - \log a) \leqslant -(\sqrt{a} - \sqrt{b})^2 + (b - a) \tag{4.7}$$

for positive a and b, the right-hand side of the last expression is bounded by

$$-\gamma^{-2} \int F_n^{i,j}(\sigma)[\sqrt{f_n^t}(\sigma^{i,j}) - \sqrt{f_n^t}(\sigma)]^2 \, d\nu_P + \gamma^{-2} \int F_n^{i,j}(\sigma)[f_n^t(\sigma^{i,j}) - f_n^t(\sigma)] \, d\nu_P. \tag{4.8}$$

Observe that for all functions h and positive integers n and m, $\langle h_{n+m+1} \rangle_{\Lambda_n^{n+m}} = h_n$. In particular, using Lemma 4.1 we have that

$$|F_n^{i,j}(\sigma) - 1| \leqslant B\gamma.$$

With this remark, and since the measure ν_P is invariant for the exclusion process, (4.8) is bounded above by

$$-\gamma^{-2}(1 - B\gamma) \int [\sqrt{f_n^t}(\sigma^{i,j}) - \sqrt{f_n^t}(\sigma)]^2 \, d\nu_P + B\gamma^{-1} \int |f_n^t(\sigma^{i,j}) - f_n^t(\sigma)| \, d\nu_P.$$

Using the elementary inequality $2ab \leqslant A^{-1}a^2 + Ab^2$, the second term of the last inequality is bounded by

$$\frac{A}{2}\gamma^{-2} I_{i,j}(f_n^t) + \frac{B}{2A} \int [\sqrt{f_n^t}(\sigma^{i,j}) + \sqrt{f_n^t}(\sigma)]^2 \, d\nu_P \leqslant \frac{A}{2}\gamma^{-2} I_{i,j}(f_n^t) + 2\frac{B^2}{A},$$

where we used in the last inequality Schwartz inequality and the fact that f_n^t is the probability density with respect to ν_P. Choosing A small enough, and taking the sum over all $i, j \in \Lambda_n$ such that $|i - j| = 1$, we get

$$\Omega_n^1 \leqslant -C_0 D_n(f_n^t) + C_0' n^d \tag{4.9}$$

for some positive constants C_0 and C_0'.

Step 2 (*bound of* Ω_n^2). Fix a bond $(i, j) \in \Lambda_n \times \Lambda_n^c$, such that $|i - j| = 1$ and decompose $\mathbb{L}_{i,j}$ into three terms,

$$\mathbb{L}_{i,j} = \mathbb{L}_{i,j}^{(0,1)} + \mathbb{L}_{i,j}^{(-1,0)} + \mathbb{L}_{i,j}^{(-1,1)}, \tag{4.10}$$

where for $(l, m) \in \{(0, 1), (-1, 0), (-1, 1)\}$, $\mathbb{L}_{i,j}^{(l,m)}$ is given by

$$(\mathbb{L}_{i,j}^{(l,m)} g)(\sigma) = r_{i,j}^{(l,m)}(\sigma) C_\gamma^{\mathrm{K},\beta}(i, j; \sigma)[g(\boldsymbol{T}_{m-l}^{j,i}\sigma) - g(\sigma)]$$

$$+ r_{j,i}^{(l,m)}(\sigma) C_\gamma^{\mathrm{K},\beta}(i, j; \sigma)[g(\boldsymbol{T}_{m-l}^{i,j}\sigma) - g(\sigma)].$$

Here for $\imath = 1, 2$, $(i, j) \in \mathbb{Z}^d \times \mathbb{Z}^d$ and a configuration σ, $\boldsymbol{T}_\imath^{i,j}\sigma$ is defined by $\boldsymbol{T}_\imath^{i,j}\sigma = \sigma - \imath\delta_i + \imath\delta_j$, and $r_{i,j}^{(l,m)}(\sigma) = 1_{\{\sigma(i)=l, \sigma(j)=m\}}$. For $k \in \mathbb{Z}^d$, δ_k is the configuration with spin 1 at site k and none elsewhere, and addition of two configurations is defined coordinate by coordinate.

The term Ω_n^2 in (4.6) can be written as a sum of terms $\Omega_{i,j}^2$ associated to the bond (i, j). The decomposition (4.10) induces an analogous decomposition for $\Omega_{i,j}^2$. We study explicitly only the one corresponding to $\mathbb{L}_{i,j}^{(-1,1)}$, that we denote by $\Omega_{i,j}^{(-1,1)}$. The other two terms are dealt with in the same way:

$$\Omega_{i,j}^{(-1,1)} = \gamma^{-2} \int f_{n+N_\gamma+1}^t(\sigma) \mathbb{L}_{i,j}^{(-1,1)} \log(f_n^t(\sigma)) \, d\nu_P.$$

Let $F_1^{i,j}$ and $F_2^{i,j}$ be defined by

$$F_1^{i,j}(\sigma) = 1_{\{\sigma(j)=1\}} C_\gamma^{\mathrm{K},\beta}(i,j;\sigma) f_{n+N_\gamma+1}^t(\sigma),$$

$$F_2^{i,j}(\sigma) = 1_{\{\sigma(j)=1\}} C_\gamma^{\mathrm{K},\beta}(i,j;\sigma^{j,i}) f_{n+N_\gamma+1}^t(\sigma^{j,i}).$$

By changing variables, $\Omega_{i,j}^{(-1,1)}$ can be rewritten as

$$\gamma^{-2} \int 1_{\{\sigma(i)=-1\}} \{\langle F_1^{i,j}(\sigma)\rangle_{\Lambda_n^{n+N_\gamma}} - \langle F_2^{i,j}(\sigma)\rangle_{\Lambda_n^{n+N_\gamma}}\} \log\left\{\frac{f_n^t(\sigma+2\delta_i)}{f_n^t(\sigma)}\right\} \mathrm{d}\nu_P(\sigma). \tag{4.11}$$

Since we have that $(a-b)(\log c - \log d)$ is negative for a, b, c and d positive real numbers, unless $a \geqslant b$ and $c \geqslant d$ or $a \leqslant b$ and $c \leqslant d$, we may introduce in the last integral the indicator function of the set $E_n^1 \cup E_n^2$, where

$$E_n^1 = \{\sigma\colon \langle F_1^{i,j}(\sigma)\rangle_{\Lambda_n^{n+N_\gamma}} \geqslant \langle F_2^{i,j}(\sigma)\rangle_{\Lambda_n^{n+N_\gamma}},\ f_n^t(\sigma+2\delta_i) \geqslant f_n^t(\sigma)\},$$

$$E_n^2 = \{\sigma\colon \langle F_1^{i,j}(\sigma)\rangle_{\Lambda_n^{n+N_\gamma}} \leqslant \langle F_2^{i,j}(\sigma)\rangle_{\Lambda_n^{n+N_\gamma}},\ f_n^t(\sigma+2\delta_i) \leqslant f_n^t(\sigma)\}.$$

We shall consider separately the integral on E_n^1 and E_n^2, and we call Ω_4 (resp. Ω_5) the integral on the set E_n^1 (resp. E_n^2). We consider first the integral on E_n^1 and rewrite it as the sum of two other terms

$$\begin{aligned}\Omega_4 = \gamma^{-2} &\int 1_{\{\sigma(i)=-1\}} \{\langle F_1^{i,j}(\sigma)\rangle_{\Lambda_n^{n+N_\gamma}} - \langle F_3^{i,j}(\sigma)\rangle_{\Lambda_n^{n+N_\gamma}}\} \\ &\times \log\left\{\frac{f_n^t(\sigma+2\delta_i)}{f_n^t(\sigma)}\right\} 1_{E_n^1}\, \mathrm{d}\nu_P(\sigma) \\ &+ \gamma^{-2} \int 1_{\{\sigma(i)=-1\}} \{\langle F_3^{i,j}(\sigma)\rangle_{\Lambda_n^{n+N_\gamma}} - \langle F_2^{i,j}(\sigma)\rangle_{\Lambda_n^{n+N_\gamma}}\} \\ &\times \log\left\{\frac{f_n^t(\sigma+2\delta_i)}{f_n^t(\sigma)}\right\} 1_{E_n^1}\, \mathrm{d}\nu_P(\sigma),\end{aligned}$$

where $F_3^{i,j}$ is defined by

$$F_3^{i,j}(\sigma) = 1_{\{\sigma(j)=1\}} C_\gamma^{\mathrm{K},\beta}(i,j;\sigma^{j,i}) f_{n+N_\gamma+1}^t(\sigma).$$

Applying Lemma 4.1, we obtain that the first line of the last expression is of order γ^{-1}. Indeed, observe that we have for all configurations σ

$$\begin{aligned}&\gamma^{-2}\langle F_1^{i,j}(\sigma)\rangle_{\Lambda_n^{n+N_\gamma}} - \langle F_3^{i,j}(\sigma)\rangle_{\Lambda_n^{n+N_\gamma}} \\ &\quad = \gamma^{-2}\langle 1_{\{\sigma(j)=1\}}[C_\gamma^{\mathrm{K},\beta}(i,j;\sigma) - C_\gamma^{\mathrm{K},\beta}(i,j;\sigma^{i,j})] f_{n+N_\gamma+1}^t(\sigma)\rangle_{\Lambda_n^{n+N_\gamma}} \\ &\quad \leqslant C_2\gamma^{-1}\langle f_{n+N_\gamma+1}^t(\sigma)\rangle_{\Lambda_n^{n+N_\gamma}} = C_2\gamma^{-1} f_n^t(\sigma)\end{aligned}$$

for some positive constant C_2, and on the set E_n^1, we have that $f_n^t(\sigma+2\delta_i) \geqslant f_n^t(\sigma)$. Hence, from inequality (4.7), the change of variables and the fact that f_n^t is a probability density with respect to ν_P, the first term of Ω_4 is bounded by

$$C_2\gamma^{-1}\int 1_{\{\sigma(i)=-1\}}\,|\,f_n^t(\sigma+2\delta_i)-f_n^t(\sigma)\,|\,\mathrm{d}\nu_P(\sigma)\leqslant C_2'\gamma^{-1}$$

for some positive constant C_2' that depends on P and β. We estimate now the second term of Ω_4. Since on E_n^1 we have $f_n^t(\sigma+2\delta_i)\geqslant f_n^t(\sigma)$, we may replace the indicator function on E_n^1 by the indicator function on the set E_n^3 defined by

$$E_n^3=\{\sigma\colon \langle F_3^{i,j}(\sigma)\rangle_{\Lambda_n^{n+N_\gamma}}\geqslant\langle F_2^{i,j}(\sigma)\rangle_{\Lambda_n^{n+N_\gamma}},\ f_n^t(\sigma+2\delta_i)\geqslant f_n^t(\sigma)\}.$$

Since for all positive x, $\log x\leqslant 2(\sqrt{x}-1)$, and since on the set E_n^3, we have that $\langle F_3^{i,j}(\sigma)\rangle_{\Lambda_n^{n+N_\gamma}}\geqslant\langle F_2^{i,j}(\sigma)\rangle_{\Lambda_n^{n+N_\gamma}}$, the second integral of Ω_4 with indicator function on E_n^3 is less than or equal to

$$2\gamma^{-2}\int 1_{\{\sigma(i)=-1\}}\,\{\langle F_3^{i,j}(\sigma)\rangle_{\Lambda_n^{n+N_\gamma}}-\langle F_2^{i,j}(\sigma)\rangle_{\Lambda_n^{n+N_\gamma}}\}$$
$$\times\left\{\frac{\sqrt{f_n^t(\sigma+2\delta_i)}}{\sqrt{f_n^t(\sigma)}}-1\right\}1_{E_n^3}\,\mathrm{d}\nu_P(\sigma).$$

By the elementary inequality $2xy\leqslant(\gamma^{-1}/\alpha)x^2+(\alpha/\gamma^{-1})y^2$ and since $(A-B)=(\sqrt{A}-\sqrt{B})(\sqrt{A}+\sqrt{B})$ we have that for every positive A,b,a,b and for every positive α,

$$2(A-B)((b/a)-1)\leqslant\frac{\gamma^{-1}}{\alpha}(\sqrt{A}-\sqrt{B})^2+\frac{\alpha}{\gamma^{-1}}(\sqrt{A}+\sqrt{B})^2((b/a)-1)^2.$$

In particular, the last integral is bounded above by

$$\frac{\gamma^{-3}}{\alpha}\int 1_{\{\sigma(i)=-1\}}\left\{\sqrt{\langle F_3^{i,j}(\sigma)\rangle_{\Lambda_n^{n+N_\gamma}}}-\sqrt{\langle F_2^{i,j}(\sigma)\rangle_{\Lambda_n^{n+N_\gamma}}}\right\}^2\mathrm{d}\nu_P(\sigma)$$
$$+\alpha\gamma^{-1}\int 1_{\{\sigma(i)=-1\}}\left\{\sqrt{\langle F_3^{i,j}(\sigma)\rangle_{\Lambda_n^{n+N_\gamma}}}+\sqrt{\langle F_2^{i,j}(\sigma)\rangle_{\Lambda_n^{n+N_\gamma}}}\right\}^2$$
$$\times\left\{\frac{\sqrt{f_n^t(\sigma+2\delta_i)}}{\sqrt{f_n^t(\sigma)}}-1\right\}^2 1_{E_n^3}\,\mathrm{d}\nu_P(\sigma). \tag{4.12}$$

The first line of this expression is bounded above by the one-bond Dirichlet form. It is equal to

$$\frac{\gamma^{-3}}{N_\gamma\alpha}\sum_{m=n+N_\gamma+1}^{n+2N_\gamma}\int\left\{\sqrt{\langle r_{i,j}^{(-1,1)}(\sigma)C_\gamma^{\mathrm{K},\beta}(i,j;\sigma^{i,j})f_m^t(\sigma^{j,i})\rangle_{\Lambda_n^m}}\right.$$
$$\left.-\sqrt{\langle r_{i,j}^{(-1,1)}(\sigma)C_\gamma^{\mathrm{K},\beta}(i,j;\sigma^{i,j})f_m^t(\sigma)\rangle_{\Lambda_n^m}}\right\}^2\mathrm{d}\nu_P(\sigma),$$

which, by Schwartz inequality, is bounded by

$$\frac{\gamma^{-3}}{N_\gamma \alpha} \sum_{m=n+N_\gamma+1}^{n+2N_\gamma} \int \langle r_{i,j}^{(-1,1)}(\sigma) C_\gamma^{\mathrm{K},\beta}(i,j;\sigma^{i,j}) \{\sqrt{f_m^t(\sigma^{j,i})} - \sqrt{f_m^t(\sigma)}\}^2 \rangle_{\Lambda_n^m} \,\mathrm{d}\nu_P(\sigma)$$

$$\leqslant C_4 \frac{\gamma^{-2}}{\alpha} \sum_{m=n+N_\gamma+1}^{n+2N_\gamma} \int \langle r_{i,j}^{(-1,1)}(\sigma) \{\sqrt{f_m^t(\sigma^{j,i})} - \sqrt{f_m^t(\sigma)}\}^2 \rangle_{\Lambda_n^m} \,\mathrm{d}\nu_P(\sigma)$$

$$= C_4 \frac{\gamma^{-2}}{\alpha} \sum_{m=n+N_\gamma+1}^{n+2N_\gamma} \int r_{i,j}^{(-1,1)}(\sigma) \{\sqrt{f_m^t(\sigma^{j,i})} - \sqrt{f_m^t(\sigma)}\}^2 \,\mathrm{d}\nu_P(\sigma),$$

where we used the fact that there exists a positive constant C_4, such that $C_\gamma^{\mathrm{K},\beta}(i,j;\sigma) \leqslant C_4$, for all configurations σ.

Finally, the second term of (4.12) is bounded by

$$4\gamma^{-1}\alpha \int 1_{\{\sigma(i)=-1\}} \frac{\langle F_3^{i,j}(\sigma)\rangle_{\Lambda_n^{n+N_\gamma}}}{f_n^t(\sigma)} \{\sqrt{f_n^t(\sigma+2\delta_i)} - \sqrt{f_n^t(\sigma)}\}^2 \mathrm{d}\nu_P(\sigma)$$

$$\leqslant 4C_4\gamma^{-1}\alpha \int 1_{\{\sigma(i)=-1\}} \{\sqrt{f_n^t(\sigma+2\delta_i)} - \sqrt{f_n^t(\sigma)}\}^2 \mathrm{d}\nu_P(\sigma) \leqslant C_5 \gamma^{-1}\alpha$$

for some positive constant C_5 that depends on P. We have changed variables and used the fact that f_n^t is a probability density with respect to ν_P.

Collecting the above inequalities, we get the following bound for Ω_4. For any positive α

$$\Omega_4 \leqslant \gamma^{-1}(C_2' + C_5\alpha) + C_4 \frac{\gamma^{-2}}{\alpha} \sum_{m=n+N_\gamma+1}^{n+2N_\gamma} \int r_{i,j}^{(-1,1)}(\sigma) \{\sqrt{f_m^t(\sigma^{j,i})} - \sqrt{f_m^t(\sigma)}\}^2 \,\mathrm{d}\nu_P(\sigma). \tag{4.13}$$

The term Ω_5 will be handled in an analogous way. It can be rewritten as

$$\Omega_5 = \gamma^{-2} \int 1_{\{\sigma(i)=-1\}} \{\langle F_2^{i,j}(\sigma)\rangle_{\Lambda_n^{n+N_\gamma}} - \langle F_4^{i,j}(\sigma)\rangle_{\Lambda_n^{n+N_\gamma}}\} \times \log\left\{\frac{f_n^t(\sigma)}{f_n^t(\sigma+2\delta_i)}\right\} 1_{E_n^2} \,\mathrm{d}\nu_P(\sigma)$$
$$+ \gamma^{-2} \int 1_{\{\sigma(i)=-1\}} \{\langle F_4^{i,j}(\sigma)\rangle_{\Lambda_n^{n+N_\gamma}} - \langle F_1^{i,j}(\sigma)\rangle_{\Lambda_n^{n+N_\gamma}}\} \times \log\left\{\frac{f_n^t(\sigma)}{f_n^t(\sigma+2\delta_i)}\right\} 1_{E_n^2} \,\mathrm{d}\nu_P(\sigma)$$

with

$$F_4^{i,j} = 1_{\{\sigma(j)=1\}} C_\gamma^{\mathrm{K},\beta}(i,j;\sigma) f_{n+N_\gamma+1}^t(\sigma^{j,i}).$$

By the same arguments used to estimate Ω_4, we obtain by exchanging the role of $f_n^t(\sigma)$ and $f_n^t(\sigma+2\delta_i)$,

$$\Omega_5 \leqslant \gamma^{-1}(C_2'+C_5\alpha)$$

$$+C_4\frac{\gamma^{-2}}{\alpha}\sum_{m=n+N_\gamma+1}^{n+2N_\gamma}\int r_{j,i}^{(-1,1)}(\sigma)\{\sqrt{f_m^t(\sigma^{i,j})}-\sqrt{f_m^t(\sigma)}\}^2\,\mathrm{d}\nu_P(\sigma)$$

for all positive α. Therefore, taking advantage of this last inequality and of (4.13), we get

$$\gamma^{-2}\int f_{n+N_\gamma+1}^t(\sigma)\mathbb{L}_{i,j}^{(-1,1)}\log(f_n^t(\sigma))\,\mathrm{d}\nu_P$$

$$\leqslant 2\gamma^{-1}(C_2'+C_5\alpha)$$

$$+C_4\frac{\gamma^{-2}}{\alpha}\sum_{m=n+N_\gamma+1}^{n+2N_\gamma}\int r_{i,j}^{(-1,1)}(\sigma)\{\sqrt{f_m^t(\sigma^{j,i})}-\sqrt{f_m^t(\sigma)}\}^2\,\mathrm{d}\nu_P(\sigma).$$

$$+C_4\frac{\gamma^{-2}}{\alpha}\sum_{m=n+N_\gamma+1}^{n+2N_\gamma}\int r_{j,i}^{(-1,1)}(\sigma)\{\sqrt{f_m^t(\sigma^{i,j})}-\sqrt{f_m^t(\sigma)}\}^2\,\mathrm{d}\nu_P(\sigma).$$

To conclude this step, we have just to sum over $\{(0,1),(-1,0),(-1,1)\}$ and over $\{(i,j)\in\Lambda_n\times\Lambda_n^c\colon |i-j|=1\}$. We obtain

$$\Omega_n^2\leqslant 6n^{d-1}\gamma^{-1}(C_2'+C_5\alpha)+C_4\frac{\gamma^{-2}}{\alpha}\sum_{m=n+N_\gamma}^{n+2N_\gamma}I_{i,j}(f_m^t) \tag{4.14}$$

for any positive α.

Step 3 (*Proof of* (4.4)). From (4.6), (4.9) and (4.14), for all positive n

$$\partial_t s_n(t)\leqslant -C_0D_n(f_n^t)+C_0'n^d+6n^{d-1}\gamma^{-1}(C_2'+C_5\alpha)$$

$$+C_4\frac{\gamma^{-2}}{\alpha}\sum_{(i,j)\in\Lambda_n\times\Lambda_n^c}\sum_{m=n+N_\gamma+1}^{n+2N_\gamma}I_{i,j}(f_m^t).$$

Multiply both sides of this inequality by $\mathrm{e}^{-n\gamma}$, sum over $1\leqslant n\leqslant M_\gamma$ and for α large enough. We get for some positive constants A_0 and A_1

$$\partial_t\mathscr{S}_M(t)\leqslant -A_0\mathscr{D}_n(f_n^t)+A_1\gamma^{-d}$$

$$+C_4'\gamma^{-2}\sum_{n=M_\gamma-2N_\gamma+1}^{M_\gamma}\mathrm{e}^{-n\gamma}\sum_{m=1}^{N_\gamma}\sum_{(i,j)\in\Lambda_n\times\Lambda_n^c}I_{i,j}(f_{m+n+N_\gamma}^t).$$

To conclude the proof of Lemma 4.3, it remains to observe that the third term on the right-hand side of the last inequality is bounded by $Const.\gamma^{-d}$. □

Corollary 4.4. *For all* $K>0$,

$$\int_0^t D_{KN_\gamma}(f_{KN_\gamma}^\tau)\,\mathrm{d}\tau\leqslant 2C_1\mathrm{e}^{K+1}\gamma^{2-d}.$$

Proof. Fix $K>0$ and $\tau\in[0,t]$, since by Schwartz inequality $n\mapsto D_n(f_n^\tau)$ is a non-decreasing function we have

$$
\begin{aligned}
D_{KN_\gamma}(f^\tau_{KN_\gamma}) &\leqslant \frac{1}{N_\gamma}\sum_{m=1}^{N_\gamma} D_{KN_\gamma+m}(f^\tau_{KN_\gamma+m})\\
&\leqslant \mathrm{e}^{K+1}\frac{1}{N_\gamma}\sum_{m=1}^{N_\gamma} D_{KN_\gamma+m}(f^\tau_{KN_\gamma+m})\,\mathrm{e}^{-(KN_\gamma+m)\gamma}\\
&\leqslant 2\mathrm{e}^{K+1}\mathscr{D}(\mu^{\mathrm{K},\beta}(\tau)|v_P). \qquad \square
\end{aligned}
$$

5. Hydrodynamic limits for Kawasaki and Glauber dynamics

In this section we prove Theorem 3.1. Let $\mathscr{C}_K(\mathbb{R}^d)$ denote the space of real continuous functions (with compact support) and denote by $\mathscr{M}$ the space of signed measures on $\mathbb{R}^d$ with total variation bounded by 1 equipped with the weak* topology induced by $\mathscr{C}_K(\mathbb{R}^d)$ via $\langle\mu,U\rangle=\int U\,\mathrm{d}\mu$.

Given a configuration $\sigma(t,.)$, we define the empirical measures $\pi^{1,\gamma}(\sigma(t,.))=\pi_t^{1,\gamma}$, and $\pi^{2,\gamma}(\sigma(t,.))=\pi_t^{2,\gamma}$ by

$$
(\pi_t^{1,\gamma},\pi_t^{2,\gamma})=\left(\gamma^d\sum_{i\in\mathbb{Z}^d}\sigma(t,i)\delta_{\gamma i},\gamma^d\sum_{i\in\mathbb{Z}^d}(\sigma(t,i))^2\delta_{\gamma i}\right),
$$

where $\delta_{\gamma i}$ is the Dirac mass at the macroscopic site γi. We shall denote in the sequel $\sigma(t,i)$ by $\sigma_t(i)$ and $(\sigma(t,i))^2$ by $\sigma(t,i)^2$.

5.1. Kawasaki dynamics

First of all, in order to prove (3.4) it is enough to show that, for any positive time t, any functions $U,\ V\in\mathscr{C}_K(\mathbb{R}^d)$ and $\delta>0$,

$$
\lim_{\gamma\to 0}\boldsymbol{P}_{\mu^\gamma}^{(\mathrm{K}),\beta}\left\{\left|\langle\pi_t^{1,\gamma},U\rangle+\langle\pi_t^{2,\gamma},V\rangle-\int_{\mathbb{R}^d}(m^{(\mathrm{K})}(t,x)U(x)\right.\right.
$$
$$
\left.\left.+\phi^{(\mathrm{K})}(t,x)V(x))\,\mathrm{d}x\right|>\delta\right\}=0,
$$

where $(m^{(\mathrm{K})}(.,.),\phi^{(\mathrm{K})}(.,.))$ is a weak solution of the hydrodynamic equations (3.5).

Fix a parameter $\beta>0$ and consider the Kawasaki dynamics at β positive. For a fixed time interval $[0,T]$, we denote by $\boldsymbol{P}_{\mu^\gamma}^{(\mathrm{K})}$ the law of the process $(\sigma_t)_{t\in[0,T]}$ accelerated by γ^{-2} on the space $D([0,T],\Omega)$ and by $Q_{\mu^\gamma}^{(\mathrm{K})}$ the law of the process $(\pi_t^{1,\gamma},\pi_t^{2,\gamma})_{t\in[0,T]}$ on the space $D([0,T],\mathscr{M}^2)$ with initial distribution μ^γ. The law of large numbers for the empirical measures $\pi_t^{1,\gamma}$ and $\pi_t^{2,\gamma}$ follows (Guo et al., 1988) from the weak convergence of the probability measures $Q_{\mu^\gamma}^{(\mathrm{K})}$ to a probability $Q^{(\mathrm{K})}$ concentrated on the deterministic trajectory $(\pi^1(t,\mathrm{d}x),\pi^2(t,\mathrm{d}x))=(m^{(\mathrm{K})}(t,x)\,\mathrm{d}x,\phi^{(\mathrm{K})}(t,x)\,\mathrm{d}x)$, where $(m^{(\mathrm{K})}(.,.),\phi^{(\mathrm{K})}(.,.))$ is a weak solution of the hydrodynamic equations (3.5). The proof of this result requires tightness, identification of the limit and uniqueness of the weak solution of the limiting equation.

Lemma 5.1 (Tightness). *The sequence* $(Q_{\mu^\gamma}^{(\mathrm{K})})_\gamma$ *is a tight family and all its limit points* Q^* *are such that*

$$Q^*\{(\pi^1,\pi^2):\ (\pi^1(t,\mathrm{d}x),\pi^2(t,\mathrm{d}x)) = (\pi^1(t,x)\,\mathrm{d}x,\pi^2(t,x)\,\mathrm{d}x)\} = 1,$$

$$Q^*\{(\pi^1,\pi^2):\ -1\leqslant\pi^1(t,x)\leqslant 1,\ 0\leqslant\pi^2(t,x)\leqslant 1\} = 1.$$

The proof of Lemma 5.1 is very simple since $\sup_\sigma \sup_i |\sigma(i)| < \infty$ and therefore is omitted.

Lemma 5.2. *All limit points* Q^* *of the sequence* $(Q_{\mu^\gamma}^{(\mathrm{K})})_\gamma$ *are concentrated on weak solutions of Eq.* (3.5).

Finally, the law of large numbers follows from the uniqueness of the weak solution of Eqs. (3.5), whose proof is given in Lemma 5.4 below.

Proof of Lemma 5.2. Denote by $\mathscr{C}_K^{1,2}([0,T]\times\mathbb{R}^d)$ the space of compact support functions $V:[0,T]\times\mathbb{R}^d\to\mathbb{R}$, twice continuously differentiable on space with continuous derivative in time. Fix a function $U=(U^1,U^2)$ such that $U^1(t,x)=U_t^1(x)$ and $U^2(t,x)=U_t^2(x)$ are in $\mathscr{C}_K^{1,2}([0,T]\times\mathbb{R}^d)$, and consider the martingale M_t^U defined by

$$M_t^U = \sum_{n=1}^{2}\left\{\langle\pi_t^{n,\gamma},U_t^n\rangle - \langle\pi_0^{n,\gamma},U_0^n\rangle - \int_0^t(\partial_s+\gamma^{-2}\mathbb{L}_\gamma^{\mathrm{K},\beta})\langle\pi_s^{n,\gamma},U_s^n\rangle\mathrm{d}s\right\}$$

with the quadratic variation $\mathscr{N}_t^U$ given by

$$\mathscr{N}_t^U = (M_t^U)^2 - \int_0^t\left\{\gamma^{-2}\mathbb{L}_\gamma^{\mathrm{K},\beta}\left(\sum_{n=1}^{2}\langle\pi_s^{n,\gamma},U_s^n\rangle\right)^2\right.$$
$$\left. -2\left(\sum_{n=1}^{2}\langle\pi_s^{n,\gamma},U_s^n\rangle\right)\gamma^{-2}\mathbb{L}_\gamma^{\mathrm{K},\beta}\left(\sum_{n=1}^{2}\langle\pi_s^{n,\gamma},U_s^n\rangle\right)\right\}\mathrm{d}s.$$

Denote by $\{e_1,\dots,e_d\}$ the orthonormal basis of $\mathbb{R}^d$, and observe that for all $i\in\mathbb{Z}^d$, $\sigma\in\Omega$ and $k=1,\dots,d$ we have $C_\gamma^{\mathrm{K},\beta}(i+e_k,i;\ \sigma)=\tau_{e_k}C_\gamma^{K,\beta}(i,i-e_k;\ \sigma)$, where τ_{e_k} is the space shift by e_k acting on Ω. Hence, a spatial summation by parts permits to rewrite the integral term of M_t^U as

$$\sum_{n=1}^{2}\int_0^t\langle\pi_s^{n,\gamma},\partial_s U_s^n\rangle\,\mathrm{d}s$$
$$-\sum_{n=1}^{2}\int_0^t\gamma^{d-1}\sum_{i\in\mathbb{Z}^d}\sum_{k=1}^{d}C_\gamma^{\mathrm{K},\beta}(i+e_k,i;\sigma_s)[\sigma_s(i+e_k)^n-\sigma_s(i)^n](\partial_k^\gamma U_s^n)(\gamma i)\,\mathrm{d}s.$$

Here ∂_k^γ represents the discrete derivative in the kth direction:

$$(\partial_k^\gamma V)(\gamma i) = \gamma^{-1}[V(\gamma(i+e_k))-V(\gamma i)].$$

Notice that the conditions imposed on Φ imply that $\Phi'(0)=-1/2$ (cf. Giacomin and Lebowitz, 1997, 1998), in particular using Lemma 4.1 and a second summation by parts, we may rewrite the second term of the last integral as

$$\sum_{n=1}^{2}\left\{\gamma^d \sum_{i\in\mathbb{Z}^d}\sum_{k=1}^{d}\int_0^t \mathrm{d}s\{(\sigma_s(i)^n)\gamma^{-2}[U_s^n(\gamma(i+e_k)) + U_s^n(\gamma(i-e_k)) - 2U_s^n(\gamma i)]\}\right\}$$

$$+\frac{\beta}{2}\sum_{n=1}^{2}\left\{\gamma^d \sum_{i\in\mathbb{Z}^d}\sum_{k=1}^{d}\int_0^t \mathrm{d}s\{(\partial_k^\gamma U_s^n)(\gamma i)(\tau_i g^n(\sigma))(\pi_s^{1,\gamma} * \partial_k^\gamma J)(\gamma i)\}\right\} + o_\gamma(1),$$

where $o_\gamma(1)$ is a random variable that converges to 0 with γ and $*$ stands for the convolution in the spatial variable. For $n=1,2$, the functions g^n are defined by

$$g^n(\sigma) = [\sigma(e_k) - \sigma(0)][\sigma(e_k)^n - \sigma(0)^n].$$

This time, however, it is not the density fields themselves that appear in the second term of the last expression but another local function of the configuration. Following the methods of Guo et al. (1988), the main step in proving the hydrodynamic equations is to replace this local function by another function of the density fields in order to close the equations. For a cylinder function Ψ, we denote its expectation with respect to the measure $v_{(m,\phi)} = v_P$ defined in (3.2) by $\tilde{\Psi}(m,\phi)$:

$$\tilde{\Psi}(m,\phi) = \int \Psi(\sigma)\,\mathrm{d}v_P(\sigma)$$

and for a positive integer ℓ and $i\in\mathbb{Z}^d$, denote the empirical mean densities on a box of size $(2\ell+1)^d$ centered at i by $(A^{1,\ell}\sigma)(i)$ and $(A^{2,\ell}\sigma)(i)$:

$$((A^{1,\ell}\sigma)(i),(A^{2,\ell}\sigma)(i)) = \left(\frac{1}{(2\ell+1)^d}\sum_{|i-j|\leqslant\ell}\sigma(j), \frac{1}{(2\ell+1)^d}\sum_{|i-j|\leqslant\ell}\sigma(j)^2\right).$$

Lemma 5.3. *For every cylinder function Ψ and every $V:[0,T]\times\mathbb{R}^d \to \mathbb{R}$ with compact support*

$$\limsup_{\varepsilon\to 0}\limsup_{\gamma\to 0} \boldsymbol{E}_{\mu^\gamma}^{(\mathrm{K})}\left[\gamma^d\sum_{i\in\mathbb{Z}^d}\int_0^T \mathrm{d}s|V_s(\gamma i)|\left\{\left|(2\varepsilon N_\gamma+1)^{-d}\sum_{|j-i|\leqslant\varepsilon N_\gamma}\tau_j\Psi(\sigma_s)\right.\right.\right.$$

$$\left.\left.\left.-\tilde{\Psi}((A^{1,\varepsilon N_\gamma}\sigma_s)(i),(A^{2,\varepsilon N_\gamma}\sigma_s)(i))\right|\right\}\right] = 0,$$

where N_γ is the integer part of γ^{-1}.

Since the support of the function V is compact, by Corollary 4.4 the proof of this lemma is very similar to the one usually used in finite volume. Nevertheless, we shall give a sketch of its proof at the end of this subsection.

Let us go on with the proof of Lemma 5.2. Now, by Lemma 5.3 and Taylor expansion applied to the functions U^1, U^2 and J, the integral term of the martingale M_t^U

can be written as

$$\sum_{n=1}^{2}\sum_{k=1}^{d}\int_0^t \mathrm{d}s\Bigg\{\langle \pi_s^{n,\gamma},(\partial_s U_s^n+\partial_k^2 U_s^n)\rangle$$

$$+\frac{\beta}{2}\sum_{i\in\mathbb{Z}^d}[(\partial_k U_s^n)(\gamma i)(\pi_s^{1,\gamma}*\partial_k J)(\gamma i)\widetilde{g^n}((\boldsymbol{A}^{1,\varepsilon N_\gamma}\sigma)(i),(\boldsymbol{A}^{2,\varepsilon N_\gamma}\sigma)(i))]\Bigg\}+o_{\gamma,\varepsilon}(1),$$

where ∂_k and ∂_k^2 represent the first and the second derivatives in the kth direction, and $o_{\gamma,\varepsilon}(1)$ is a random variable that converges to 0 when $\gamma\to 0$ and $\varepsilon\to 0$.

Moreover, remark that

$$\widetilde{g^1}((\boldsymbol{A}^{1,\varepsilon N_\gamma}\sigma)(i),(\boldsymbol{A}^{2,\varepsilon N_\gamma}\sigma)(i))=2[(\boldsymbol{A}^{2,\varepsilon N_\gamma}\sigma)(i)-((\boldsymbol{A}^{1,\varepsilon N_\gamma}\sigma)(i))^2],$$

$$\widetilde{g^2}((\boldsymbol{A}^{1,\varepsilon N_\gamma}\sigma)(i),(\boldsymbol{A}^{2,\varepsilon N_\gamma}\sigma)(i))=2(\boldsymbol{A}^{1,\varepsilon N_\gamma}\sigma)(i)[1-(\boldsymbol{A}^{2,\varepsilon N_\gamma}\sigma)(i)].$$

On the other hand, a simple computation shows that the quadratic variation of the martingale M_t^U is equal to

$$\mathcal{N}_t^U=(M_t^U)^2-\gamma^{2d}\sum_{\substack{i,j\\|i-j|=1}}\int_0^t\Bigg\{C_\gamma^{\mathrm{K},\beta}(i,j;\sigma_s)$$

$$\times\left(\sum_{n=1}^{2}\gamma^{-1}[U_s^n(\gamma i)-U_s^n(\gamma j)][\sigma_s(i)^n-\sigma_s(j)^n]\right)^2\Bigg\}\mathrm{d}s$$

and then it vanishes as γ goes to 0. By Doob's inequality, for every $\delta>0$,

$$\lim_{\gamma\to 0}\boldsymbol{P}_{\mu^\gamma}^{(\mathrm{K})}\left\{\sup_{0\leqslant t\leqslant T}|M_t^U|>\delta\right\}=0.$$

Therefore, collecting the above arguments and using Lemma 5.1, we obtain that any limit point Q^* of the sequence $(Q_{\mu^\gamma}^{(\mathrm{K})})_\gamma$ is such that

$$Q^*\Bigg\{(m,\phi)\colon\Bigg|\langle m_T,U_T\rangle-\langle m_0,U_0\rangle-\int_0^T\langle m_s,(\partial_s U_s+\Delta U_s)\rangle\,\mathrm{d}s$$

$$+\langle\phi_T,V_T\rangle-\langle\phi_0,V_0\rangle-\int_0^T\langle\phi_s,(\partial_s V_s+\Delta V_s)\rangle\,\mathrm{d}s$$

$$-\beta\sum_{k=1}^{d}\int_0^T\langle(m_s*(\partial_k J))((\phi_s*\alpha_\varepsilon)-(m_s*\alpha_\varepsilon)^2),(\partial_k U_s)\rangle\,\mathrm{d}s$$

$$-\beta\sum_{k=1}^{d}\int_0^T\langle(m_s*(\partial_k J))(m_s*\alpha_\varepsilon)(1-(\phi_s*\alpha_\varepsilon)),(\partial_k V_s)\rangle\mathrm{d}s\Bigg|>\delta\Bigg\}=0$$

for any $\delta>0$ and $U,V\in\mathscr{C}_K^{1,2}([0,T]\times\mathbb{R}^d)$, where α_ε is defined as $\alpha_\varepsilon(x)=(2\varepsilon)^{-d}1_{[-\varepsilon,\varepsilon]^d}(x)$. Let ε tend to 0 and by arbitrariness of δ we obtain the statement of Lemma 5.2. □

Proof of Lemma 5.3. To simplify the notation, for $\sigma \in \Omega$, denote by $\boldsymbol{V}^{N_\gamma,\varepsilon}(\sigma)$ the expression

$$\boldsymbol{V}^{N_\gamma,\varepsilon}(\sigma) = \frac{1}{(2\varepsilon N_\gamma + 1)^d} \left| \sum_{|j| \leqslant \varepsilon N_\gamma} \Psi(\sigma) - \tilde{\Psi}((\boldsymbol{A}^{1,\varepsilon N_\gamma}\sigma)(0), (\boldsymbol{A}^{2,\varepsilon N_\gamma}\sigma)(0)) \right|.$$

Fix $M > 0$ such that $[-M, M]^d$ contains the support of V_s for all $s \in [0, T]$. We have

$$\boldsymbol{E}_{\mu^\gamma}^{(\mathrm{K})} \left[\gamma^d \sum_{i \in \mathbb{Z}^d} \int_0^T |V_s(\gamma i)| \tau_i \boldsymbol{V}^{N_\gamma,\varepsilon}(\sigma_s)\, \mathrm{d}s \right]$$

$$\leqslant \|V\|_\infty \boldsymbol{E}_{\mu^\gamma}^{(\mathrm{K})} \left[\gamma^d \sum_{i \in \Lambda_{MN_\gamma}} \int_0^T \tau_i \boldsymbol{V}^{N_\gamma,\varepsilon}(\sigma_s)\, \mathrm{d}s \right].$$

Denote $\bar{f}^T = (1/T) \int_0^T f^s_{(M+2)N_\gamma}\, \mathrm{d}s$, where $f^s_{(M+2)N_\gamma}$ is the probability density with respect to $\nu_{P,(M+2)N_\gamma} = \nu_P^{M,N_\gamma}$ of the restriction of the measure $\mu^{\mathrm{K},\beta}(s)$ to the box $\Lambda_{(M+2)N_\gamma}$ (see Section 3). Since the function $\sum_{i \in \Lambda_{MN_\gamma}} \tau_i \boldsymbol{V}^{N_\gamma,\varepsilon}(\sigma)$ depends on the configuration σ only through the variables $\{\sigma(k) : k \in \Lambda_{(M+1)N_\gamma}\}$, by Fubini's theorem, Dirichlet form convexity and Corollary 4.4, there exists a positive constant C that depends on M, β and P such that the right-hand side of the last inequality is bounded by

$$\left\{ T\|V\|_\infty \int \gamma^d \sum_{i \in \Lambda_{MN_\gamma}} \tau_i \boldsymbol{V}^{N_\gamma,\varepsilon}(\sigma) \bar{f}^T(\sigma)\, \mathrm{d}\nu_P^{M,N_\gamma}(\sigma) - A\gamma^{d-2} D_{(M+2)N_\gamma}(\bar{f}^T) \right\} + AC$$

for all positive A. It follows that, in order to prove Lemma 5.3 it is enough to show that for each positive A

$$\limsup_{\varepsilon \to 0} \limsup_{\gamma \to 0} \sup \left\{ \int \gamma^d \sum_{i \in \Lambda_{MN_\gamma}} \tau_i \boldsymbol{V}^{N_\gamma,\varepsilon}(\sigma) f(\sigma)\, \mathrm{d}\nu_P^{M,N_\gamma}(\sigma) \right.$$

$$\left. - A\gamma^{d-2} D_{(M+2)N_\gamma}(f) \right\} = 0,$$

where the supremum is carried over all probability densities f with respect to ν_P^{M,N_γ}.

The proof of this limit relies on the usual one and two blocks estimates (cf. Guo et al., 1988; Kipnis et al., 1989) and therefore is omitted. □

Lemma 5.4 (Uniqueness). *For any $T > 0$, Eq.* (3.5) *has a unique weak solution in the class $L^\infty([0, T]) \times \mathbb{R}^d) \times L^\infty([0, T]) \times \mathbb{R}^d)$.*

Proof. The proof follows the arguments in Giacomin and Lebowitz (1997, 1998) adapted to the infinite volume case. For a positive time $t > 0$, $f \in \mathscr{C}_K(\mathbb{R}^d)$ and $\varepsilon > 0$, let $\mathscr{H}^f_{t,\varepsilon} : [0, t] \times \mathbb{R}^d \to \mathbb{R}$ be defined by

$$\mathscr{H}^f_{t,\varepsilon}(s, x) = (f * h_{t+\varepsilon-s})(x),$$

where $h_{t+\varepsilon-s}(.)$ is the heat kernel given by

$$h_{t+\varepsilon-s}(x) = (2\pi(t+\varepsilon-s))^{-d/2}\exp\left\{-\frac{1}{4(t+\varepsilon-s)}\sum_{k=1}^{d}(x_k)^2\right\}.$$

In the appendix it is proven that $\mathscr{H}^f_{t,\varepsilon}$ solves the equation $\partial_t\rho = \Delta\rho$ on $[0,t]\times\mathbb{R}^d$ and that

$$\sum_{k=1}^{d}\int_0^t |\langle\partial_k\mathscr{H}^f_{t,\varepsilon}(s,.)\rangle|\,\mathrm{d}s \leqslant C_1(\sqrt{t+\varepsilon}-\sqrt{\varepsilon})||f||_1$$
$$\leqslant C_1\sqrt{t}||f||_1, \tag{5.1}$$

where C_1 is a positive constant that depends on d and ∂_k is the first derivative in the kth direction.

Let us consider (m,ϕ) and $(\tilde{m},\tilde{\phi})$ two weak solutions of (3.5) with the same initial datum. Set $\bar{m} = m-\tilde{m}$, $\bar{\phi} = \phi-\tilde{\phi}$ and $W = |\bar{m}|+|\bar{\phi}|$. To keep the notation simple, for $(s,x)\in\mathbb{R}_+\times\mathbb{R}^d$, we shall denote $\bar{m}_s(x) = \bar{m}(s,x)$ and $\bar{\phi}_s = \bar{\phi}(s,x)$. For $1\leqslant k\leqslant d$ and $s>0$, let M_k and F_k be defined by

$$M_k(m_s,\phi_s) = (\phi_s - m_s^2)(\partial_k J * m_s),$$

$$F_k(m_s,\phi_s) = m_s(1-\phi_s)(\partial_k J * m_s).$$

Observe that for all $s\in[0,t]$,

$$M_k(m_s,\phi_s) - M_k(\tilde{m}_s,\tilde{\phi}_s) = (\partial_k J * \bar{m}_s)(\tilde{\phi}_s - \tilde{m}_s^2) + (\partial_k J * m_s)[\bar{\phi}_s - \bar{m}_s(m_s+\tilde{m}_s)]$$

and

$$F_k(m_s,\phi_s) - F_k(\tilde{m}_s,\tilde{\phi}_s) = (\partial_k J * \bar{m}_s)m_s(1-\phi_s) + (\partial_k J * \tilde{m}_s)[\bar{m}_s(1-\phi_s) - \tilde{m}_s\bar{\phi}_s].$$

It follows that there exists a positive constant C_2 that depends on $||m||_\infty$, $||\phi||_\infty$ and $\sup_{1\leqslant k\leqslant d}||\partial_k J||$ such that, for almost every $(s,x)\in[0,t]\times\mathbb{R}^d$,

$$|M_k(m_s,\phi_s) - M_k(\tilde{m}_s,\tilde{\phi}_s)| \leqslant C_2 R(t),$$

$$|F_k(m_s,\phi_s) - F_k(\tilde{m}_s,\tilde{\phi}_s)| \leqslant C_2 R(t).$$

Here $R(t)$ stands for the essential sup of W in $[0,t]\times\mathbb{R}^d$:

$$R(t) = \operatorname{ess}\sup_{[0,t]\times\mathbb{R}^d}(W(s,x)).$$

Since (m,ϕ) and $(\tilde{m},\tilde{\phi})$ are two weak solutions of (3.5), we obtain by (5.1) that for all $0\leqslant\tau\leqslant t$

$$|\langle\bar{m}(\tau,.),\mathscr{H}^f_{\tau,\varepsilon}(\tau,.)\rangle| = \beta\left|\sum_{k=1}^{d}\int_0^\tau\langle(M_k(m_s,\phi_s) - M_k(\tilde{m}_s,\tilde{\phi}_s)),\partial_k\mathscr{H}^f_{\tau,\varepsilon}(s,.)\rangle\,\mathrm{d}s\right|,$$
$$\leqslant C_3\sqrt{t}R(t)||f||_1,$$

$$|\langle\bar{\phi}(\tau,.),\mathscr{H}^f_{\tau,\varepsilon}(\tau,.)\rangle| = \beta\left|\sum_{k=1}^{d}\int_0^\tau\langle(F_k(m_s,\phi_s) - F_k(\tilde{m}_s,\tilde{\phi}_s)),\partial_k\mathscr{H}^f_{\tau,\varepsilon}(s,.)\rangle\,\mathrm{d}s\right|,$$
$$\leqslant C_3\sqrt{t}R(t)||f||_1,$$

for some positive constant C_3. By observing that h_ε is an approximate identity in ε, we obtain that

$$|\langle \bar{m}(\tau,.), f\rangle| \leqslant C_3\sqrt{t}R(t)||f||_1,$$

$$|\langle \bar{\phi}(\tau,.), f\rangle| \leqslant C_3\sqrt{t}R(t)||f||_1$$

for all $f \in \mathscr{C}_K(\mathbb{R}^d)$ and then for all $f \in L^1(\mathbb{R}^d)$. It follows that, for $0\leqslant\tau\leqslant t$ and $f \in L^1(\mathbb{R}^d)$,

$$\langle |\bar{m}(\tau,.)|, f\rangle \leqslant C_3\sqrt{t}R(t)||f||_1,$$

$$\langle |\bar{\phi}(\tau,.)|, f\rangle| \leqslant C_3\sqrt{t}R(t)||f||_1.$$

Therefore, for all $0\leqslant\tau\leqslant t$ and $f \in L^1(\mathbb{R}^d)$,

$$\langle W(\tau,.), f\rangle \leqslant 2C_3\sqrt{t}R(t)||f||_1, \tag{5.2}$$

which implies that, for all $\tau \in [0,t]$ (see the appendix),

$$W(\tau,.) \in L^\infty(\mathbb{R}^d) \quad \text{and} \quad ||W(\tau,.)||_\infty \leqslant 2C_3\sqrt{t}R(t). \tag{5.3}$$

On the other hand, proceeding as in the proof of (5.2), we obtain

$$\langle W(\tau,.), f\rangle \leqslant 2C_3\sqrt{t}\tilde{R}(t),$$

where $\tilde{R}(t)$ is given by

$$\tilde{R}(t) = \sup_{0\leqslant s\leqslant t} ||W(s,.)||_\infty.$$

This implies that

$$\tilde{R}(t) \leqslant 2C_3\sqrt{t}\tilde{R}(t).$$

Choosing $t=t_0$ such that $2C_3\sqrt{t_0} < 1$, this gives uniqueness in $[0,t_0]\times\mathbb{R}^d$. To conclude the proof we have just to repeat the same arguments in $[t_0, 2t_0]$, and in each interval $[kt_0, (k+1)t_0]$, $k \in N$, $k > 1$. $\square$

5.2. Glauber dynamics

The proof of the hydrodynamical limit in the Glauber case is based on martingales arguments and does not require Dirichlet form estimates. Following the same strategy as in the Kawasaki case we provide the analogues of Lemmas 5.1, 5.2 and 5.4. Fixing a parameter $\beta > 0$ and the time interval $[0,T]$, we denote by $\boldsymbol{P}^{(\mathrm{G})}_{\mu^\gamma}$ the law of the process $(\sigma(t,.))_{t\in[0,T]}$ (without acceleration) on the space $D([0,T],\Omega)$ and by $Q^{(\mathrm{G})}_{\mu^\gamma}$ the law of the process $(\pi_t^{1,\gamma}, \pi_t^{2,\gamma})_{t\in[0,T]}$ on the space $D([0,T],\mathscr{M}^2)$ with initial distribution μ^γ. As in the Kawasaki case the proof of the analogue of Lemma 5.1 is simple and is omitted. We shall give only the proof of the analogue of Lemma 5.2: All limit points Q^* of the sequence $(Q^{(\mathrm{G})}_{\mu^\gamma})_\gamma$ are concentrated on weak solutions of Eq. (3.7). Finally, the proof of the uniqueness of Eq. (3.7) goes on along the same lines as in the Kawasaki case. It is easier, since we do not need to use properties of the heat kernel.

Denote by $\mathscr{C}_K^{1,0}([0,T]\times\mathbb{R}^d)$ the space of continuous functions with compact support and with derivative continuous in time. Let $U=(U^1,U^2)\in\mathscr{C}_K^{1,0}([0,T]\times\mathbb{R}^d)\times\mathscr{C}_K^{1,0}([0,T]\times\mathbb{R}^d)$, and consider the martingale $M_t^{(\mathrm{G}),U}$ defined by

$$M_t^{(\mathrm{G}),U}=\sum_{\iota=1}^{2}\left\{\langle\pi_t^{\iota,\gamma},U_t^\iota\rangle-\langle\pi_0^{\iota,\gamma},U_0^\iota\rangle-\int_0^t(\partial_s+\mathbb{L}_\gamma^{\mathrm{G},\beta})\langle\pi_s^{\iota,\gamma},U_s^\iota\rangle\,\mathrm{d}s\right\}.$$

Observe that since for all $\sigma\in\Omega$ and $i\in\mathbb{Z}^d$, $\sigma(i)\in\{-1,0,1\}$ we have $1_{\{\sigma(i)=-1\}}=(1/2)\sigma(i)(\sigma(i)-1), 1_{\{\sigma(i)=0\}}=(1-(\sigma(i))^2)$ and $1_{\{\sigma(i)=1\}}=(1/2)\sigma(i)(\sigma(i)+1)$ we may thus rewrite the generator $\mathbb{L}_\gamma^{(\mathrm{G}),\beta}$ as

$$\mathbb{L}_\gamma^{(\mathrm{G}),\beta}=\mathbb{L}^{(f)}+\mathbb{L}^-+\mathbb{L}^+,$$

with

$$(\mathbb{L}^{(f)}g)(\sigma)=\frac{1}{2}\sum_{i\in\mathbb{Z}^d}\frac{\mathrm{e}^{-(\beta/2)(\nabla_i^{(f)}H_\gamma)(\sigma)}}{2\cosh(\beta/2(\nabla_i^{(f)}H_\gamma))}[(\nabla_i^{(f)}g)(\sigma)],$$

$$(\mathbb{L}^{\mp}g)(\sigma)=\frac{1}{2}\sum_{i\in\mathbb{Z}^d}\frac{\mathrm{e}^{-(\beta/2)(\nabla_i^{\mp}H_\gamma)(\sigma)}}{2\cos\cosh(\beta/2(\nabla_i^{\mp}H_\gamma))}\left((1-\sigma(i)^2)+\frac{\sigma(i)(\sigma(i)\pm1)}{2}\right)$$
$$\times[(\nabla_i^{\mp}g)(\sigma)],$$

where for a cylinder function $F,(\nabla_i^{(f)}F)(\sigma)$ is defined by

$$(\nabla_i^{(f)}F)(\sigma)=F(\sigma^{(f),i})-F(\sigma).$$

For $i\in\mathbb{Z}^d,\sigma^{(f),i}$ is a configuration obtained from σ by flipping the value at i

$$(\sigma^{(f),i})(l)=\begin{cases}\sigma(l) & \text{if } l\neq i,\\ -\sigma(i) & \text{if } l=i.\end{cases}$$

On the other hand, for all $\sigma\in\Omega$ and $i\in\mathbb{Z}^d$ we have

$$\exp\left(-\frac{\beta}{2}(\nabla_i^{(f)}H_\gamma)(\sigma)\right)=\exp(-\beta\sigma(i)\alpha^{h_1}(\gamma i))\exp(2\gamma^dJ_\gamma(0)),$$

$$\exp\left(-\frac{\beta}{2}(\nabla_i^{\mp}H_\gamma)(\sigma)\right)=\exp\left(\mp\frac{\beta}{2}[\alpha^{h_1}(\gamma i)+\tilde{h}_2^\gamma(2\sigma(i)\mp1)]\right)\exp(-\gamma^dJ_\gamma(0)),$$

where α^{h_1} and $\tilde{h}_2^\gamma$ are defined by

$$\alpha^{h_1}(\gamma i)=2(\pi^{1,\gamma}*J)(\gamma i)+h_1,\quad \tilde{h}_2^\gamma=\left(h_2-\gamma^d\sum_{k\in\mathbb{Z}^d}J(\gamma k)\right);$$

for $s\in[0,T]$, we shall denote

$$\alpha_s^{h_1}(\gamma i)=2(\pi_s^{1,\gamma}*J)(\gamma i)+h_1.$$

By using the relation $\mathrm{e}^b=(\cosh(b)+\sinh(b))$ it is easy to show that the martingale $M_t^{(\mathrm{G}),U}$ can be written as

$$M_t^{(\mathrm{G}),U}=M_t^1+M_t^2+o_\gamma(1),$$

where M_t^1 and M_t^2 are the martingales given by

$$M_t^1 = \langle \pi_t^{1,\gamma}, U_t^1 \rangle - \langle \pi_0^{1,\gamma}, U_0^1 \rangle - \int_0^t \langle \pi_s^{1,\gamma}, \partial_s U_s^1 \rangle \, \mathrm{d}s$$

$$-\frac{1}{2}\int_0^t \{\langle -\pi_s^{1,\gamma}, U_s^1 \rangle + \langle \pi_s^{2,\gamma}, U_s^1(.)\tanh(\beta\alpha_s^{h_1}(.))\rangle\} \, \mathrm{d}s$$

$$-\frac{1}{4}\gamma^d \sum_{i\in\mathbb{Z}^d} \int_0^t U_s^1(\gamma i)\left\{\tanh\left(\frac{\beta}{2}[\alpha_s^{h_1}(\gamma i)+\tilde{h}_2^\gamma]\right) + \tanh\left(\frac{\beta}{2}[\alpha_s^{h_1}(\gamma i)-\tilde{h}_2^\gamma]\right)\right\} \mathrm{d}s$$

$$-\frac{1}{8}\int_0^t \left\langle \pi_s^{1,\gamma}, U_s^1(.)\left\{\tanh\left(\frac{\beta}{2}[\alpha_s^{h_1}(.)+\tilde{h}_2^\gamma]\right)\right.\right.$$

$$\left.\left. -\tanh\left(\frac{\beta}{2}[\alpha_s^{h_1}(.)-\tilde{h}_2^\gamma]\right) - 2\right\}\right\rangle \mathrm{d}s$$

$$-\frac{1}{8}\int_0^t \left\langle -\pi_s^{2,\gamma}, U_s^1(.)\left\{\tanh\left(\frac{\beta}{2}[\alpha_s^{h_1}(.)+\tilde{h}_2^\gamma]\right) + \tanh\left(\frac{\beta}{2}[\alpha_s^{h_1}(.)-\tilde{h}_2^\gamma]\right)\right\}\right\rangle \mathrm{d}s$$

and

$$M_t^2 = \langle \pi_t^{2,\gamma}, U_t^2 \rangle - \langle \pi_0^{2,\gamma}, U_0^2 \rangle - \int_0^t \langle \pi_s^{2,\gamma}, \partial_s U_s^2 \rangle \, \mathrm{d}s$$

$$-\frac{1}{4}\gamma^d \sum_{i\in\mathbb{Z}^d} \int_0^t U_s^2(\gamma i)\left\{\tanh\left(\frac{\beta}{2}[\alpha_s^{h_1}(\gamma i)+\tilde{h}_2^\gamma]\right)\right.$$

$$\left. -\tanh\left(\frac{\beta}{2}[\alpha_s^{h_1}(\gamma i)-\tilde{h}_2^\gamma]\right) + 2\right\} \mathrm{d}s$$

$$-\frac{1}{8}\int_0^t \left\langle -\pi_s^{2,\gamma}, U_s^2(.)\left\{\tanh\left(\frac{\beta}{2}[\alpha_s^{h_1}(.)+\tilde{h}_2^\gamma]\right)\right.\right.$$

$$\left.\left. -\tanh\left(\frac{\beta}{2}[\alpha_s^{h_1}(.)-\tilde{h}_2^\gamma]\right) + 6\right\}\right\rangle \mathrm{d}s$$

$$-\frac{1}{8}\int_0^t \left\langle \pi_s^{1,\gamma}, U_s^2(.)\left\{\tanh\left(\frac{\beta}{2}[\alpha_s^{h_1}(.)+\tilde{h}_2^\gamma]\right) + \tanh\left(\frac{\beta}{2}[\alpha_s^{h_1}(.)-\tilde{h}_2^\gamma]\right)\right\}\right\rangle \mathrm{d}s.$$

On the other hand, a simple computation shows that the quadratic variation $\mathcal{N}^{(\mathrm{G}),1}$ (resp. $\mathcal{N}^{(\mathrm{G}),2}$) of the martingale M_t^1 (resp. M_t^2) vanishes as $\gamma \to 0$. Therefore, using Chebychev's inequality and Doob's inequality, we obtain

$$\lim_{\gamma\to 0} \boldsymbol{P}_{\mu^\gamma}^{(\mathrm{G})}\left\{\left[\sup_{0\leqslant s\leqslant T}|M_s^1| + \sup_{0\leqslant s\leqslant T}|M_s^2|\right] > \delta\right\} = 0$$

for any positive δ.

To conclude the proof we have just to let $\gamma \to 0$ and to follow the same arguments as in the Kawasaki case. □

6. Non-gradient dynamics

In this section, we consider a different kind of dynamics reversible for the Gibbs measure associated to the Hamiltonian (1.1) (with $h_2 = 1$) which is of the so-called non-gradient type (Kipnis and Landim, 1999). We consider a system of N spins on a d-dimensional torus $\mathbb{T}^d$. At times exponentially distributed each bond $(i,j) \in \mathbb{T}^d \times \mathbb{T}^d$, $|i-j| = 1$ changes its configuration $(\sigma(i), \sigma(j))$ independently of the others (or stays unchanged) to the new configuration $(\sigma'(i), \sigma'(j))$ in such a way that $|\sigma(i) - \sigma(i)'| = 1$, $|\sigma(j) - \sigma'(j)| = 1$ and $\sigma(i) + \sigma(j) = \sigma'(i) + \sigma'(j)$ with jump rates chosen to satisfy the detailed balance condition with respect the Hamiltonian

$$H_\gamma(\eta) = - \sum_{i,j \in \mathrm{T}^d} J_\gamma(i-j)\sigma(i)\sigma(j).$$

In other words, the transitions allowed for a bond (i,j) are

$$(0,-1) \Leftrightarrow (-1,0), \quad (1,0) \Leftrightarrow (0,1),$$

$$(1,-1) \Leftrightarrow (0,0), \quad (0,0) \Leftrightarrow (-1,1).$$

We remark that the difference between the number of positive and negative spins is conserved by this dynamics, while the number of zero spins is not, because negative and positive neighbouring spins can annihilate to create two spins with zero value or vice versa two zero spins can disappear to give rise to a couple of spins ± 1.

This dynamics, when reformulated as a lattice gas, turns out to be at $\beta = 0$ the generalized exclusion process introduced in Kipnis et al. (1994). To match the notations in that paper we prefer to use in this section the representation of the system in terms of the occupation number $\eta(i) = 0,1,2$ instead of the spin variable $\sigma(i) = -1,0,1$, their relation being $\sigma(i) = \eta(i) - 1$. In each site of the torus $\mathbb{T}^d$ there are at most two particles. A configuration of the system is an element η of $\mathbb{X}_N = \{0,1,2\}^{\mathbb{T}^d}$, where N is the number of sites in $\mathbb{T}^d$. Particles move on the torus in the following way. A particle in i jumps with a given rate to the nearest neighbour j if in j there is at most one particle. We call $\eta^{i,j}$ the configuration obtained from η letting one particle jump from i to j:

$$(\eta^{i,j})(k) = \begin{cases} \eta(k) & \text{if } k \neq i,j, \\ \eta(k) - 1 & \text{if } k = i, \\ \eta(k) + 1 & \text{if } k = j. \end{cases}$$

For $(i,j) \in \mathbb{T}^d$ and every cylindrical function $F : \mathbb{X}_N \to \mathbb{R}$, define $(\nabla_{i,j}F)(\eta)$ by

$$(\nabla_{i,j}F)(\eta) = r_{i,j}(\eta)\{F(\eta^{i,j}) - F(\eta)\},$$

where

$$r_{i,j}(\eta) = 1\{\eta(i) > 0,\ \eta(j) < 2\},$$

The jump rates are

$$C_\gamma^\beta(i,j;\eta) = \Phi\{\beta(H_\gamma(\eta^{i,j}) - H_\gamma(\eta))\},$$

with

$$H_\gamma(\eta) = -\sum_{i,j\in\mathbb{T}^d} J_\gamma(i-j)\eta(i)\eta(j), \tag{6.1}$$

where J_γ is the Kac potential defined in Section 2 and $\Phi : \mathbb{R} \to \mathbb{R}_+$ is a continuously differentiable function in a neighbourhood of 0, such that $\Phi(0) = 1$, satisfying the detailed balance condition

$$\Phi(E) = \exp(-E)\Phi(-E). \tag{6.2}$$

The generator of this jump Markov process $(\eta_t)_{t\geqslant 0}$ is given by

$$(\mathbb{L}_\gamma^\beta f)(\eta) = (1/2) \sum_{\substack{i,j\in\mathbb{T}^d\\ |i-j|=1}} C_\gamma^\beta(i,j;\eta)\{(\nabla_{i,j} f)(\eta)\},$$

where we have made explicit the dependence on the parameter $\beta\geqslant 0$.

Lemma 6.9 shows that the dynamics with parameter $\beta > 0$ is a weak perturbation of the generalized simple exclusion process GSEP in Kipnis et al. (1994) and reduces to it at $\beta = 0$. We shall denote the generator of GSEP by $\mathbb{L}_\gamma^0$.

For $\varphi\geqslant 0$, define $\bar{v}_\varphi^N$ as the product measure on $\mathbb{X}_N$ with marginals given by

$$\bar{v}_\varphi^N\{\eta(0) = r\} = \frac{\varphi^r}{1+\varphi+\varphi^2}, \quad r = 0, 1, 2.$$

Let $R(\varphi)$ be the mean occupation number of particles under $\bar{v}_\varphi^N$:

$$R(\varphi) = E_{\bar{v}_\varphi^N}[\eta(0)].$$

The function $R : \mathbb{R}_+ \to [0, 2)$ is a bijection and we denote by $\psi : [0, 2) \to \mathbb{R}_+$ its inverse. For every α in $[0, 2)$, we denote by v_α^N the product measure $\bar{v}_{\psi(\alpha)}^N$ so that the density of particles on each site is α:

$$E_{v_\alpha^N}[\eta(x)] = \alpha \quad \text{for } x \text{ in } \mathbb{X}_N.$$

We will use the notation v_α for the product measure on the infinite volume product space $\mathbb{X} = \{0, 1, 2\}^{\mathbb{Z}^d}$ and $\langle f\rangle_\alpha$ for the expectation of a cylinder function f with respect to v_α or v_α^N:

$$\langle f\rangle_\alpha = \int f(\eta)v_\alpha(\mathrm{d}\eta). \tag{6.3}$$

The one-parameter family $(v_\alpha^N)_\alpha$ of probability measures is reversible for the generator $\mathbb{L}_\gamma^0$ (GSEP) and for $\beta > 0$ the one-parameter family of probability measures $(v_\alpha^{\beta,N})_\alpha$ given by

$$v_\alpha^{\beta,N}(\eta) = \frac{\exp\{-\beta H_\gamma(\eta)\}}{Z_\alpha^\beta} v_\alpha^N(\eta)$$

is reversible for the dynamics with $\beta > 0$. Here Z_α^β is the normalization constant.

We now choose $N = \gamma^{-1}$ and speed up the generator as γ^{-2}, as in the Kawasaki case, and study the limit $N \to \infty$. In this section we show that, starting from a sequence of measures on $\mathbb{X}_N$ associated to the same initial profile ρ_0, the density field converges, as N increases to infinity, to the weak solution of the non-linear parabolic Equation (6.5), where the diffusion matrix D is given (Kipnis et al., 1994) by (6.4) below.

It has been proved in Kipnis et al. (1994) that the coefficients $D_{k,m}(\rho)$ are non-linear continuous functions of ρ and that D is strictly elliptic. That is not enough to prove the uniqueness of weak solutions of (6.5), which is easy to prove instead if the diffusion coefficient is known to be locally Lipschitz continuous (for example by the method in Landim et al. (1998)).

In order to define the diffusion coefficient, we need to establish some notation and to consider the generalized exclusion process in the infinite volume space $\mathbb{X}$.

For i in $\mathbb{Z}^d$, let τ_i denote the space shift by i units on $\mathbb{X}$. For a cylinder function F on $\mathbb{X}$, define the formal sum

$$\Gamma_F(\eta) = \sum_{j\in\mathbb{Z}^d} (\tau_j F)(\eta)$$

which does not make sense but for which the quantities $\{\nabla_{0,e_k}\Gamma_F, 1\leqslant k\leqslant d\}$ are well defined. Here $\{e_1,\dots,e_d\}$ are the unitary vectors in the coordinate directions of $\mathbb{Z}^d$. For each α in $[0,2]$, let $D(\alpha) = \{D_{k,m}(\alpha),\ 1\leqslant k,\ m\leqslant d\}$ be the symmetric matrix defined by the following variational formula:

$$a\cdot D(\alpha)a = \frac{1}{2\chi(\alpha)} \inf_F \sum_{k=1}^{d} \langle (a_k r_{0,e_k} + \nabla_{0,e_k}\Gamma_F)^2\rangle_\alpha \tag{6.4}$$

for any vector a in $\mathbb{R}^d$. In this formula $\chi(\alpha)$ is the static compressibility defined by

$$\chi(\alpha) = \langle \eta(0)^2\rangle_\alpha - \langle \eta(0)\rangle_\alpha^2 = \langle \sigma(0)^2\rangle_\alpha - \langle \sigma(0)\rangle_\alpha^2.$$

For a measure μ on $\mathbb{X}_N$, denote by P_μ the probability measure on the path space $D(\mathbb{R}_+,\mathbb{X}_N)$ corresponding to the Markov process (η_t) with generator speeded up by N^2 and starting from μ, and by $\boldsymbol{E}_{P_\mu}$ the expectation with respect to P_μ.

Let $\mathscr{M} = \mathscr{M}(\mathbb{T}^d)$ be the space of positive measures on the d-dimensional torus $\mathbb{T}^d$ with total mass bounded by $2d$. For each configuration η, denote by $\pi^N = \pi^N(\eta)$ the positive measure obtained assigning mass N^{-d} to each particle of η:

$$\pi^N = N^{-d}\sum_{j\in\mathrm{T}^d} \eta(j)\delta_{j/N},$$

where δ_x is the Dirac measure concentrated on x. For each $t\geqslant 0$, denote by $\pi_t = \pi_t^N$ the empirical measure at time $t:\pi_t = \pi^N(\eta_t)$. For a continuous function U and π in $\mathscr{M}(\mathbb{T}^d)$, we shall denote by $\langle \pi, U\rangle$ the integral of the function U with respect to the measure π.

Fix $T>0$. For each probability measure μ on $\mathbb{X}_N$, denote by Q_μ^N the measure on the state space $D([0,T],\mathscr{M})$ induced by the Markov process π_t speeded up by N^2 and μ^N.

Theorem 6.1. *Consider a sequence of probability measures μ^N on $\mathbb{X}_N$ associated to the initial profile ρ_0 in the following sense*:

$$\lim_{N\to\infty} \mu^N\{|\langle \pi^N(\eta), U\rangle - \langle \rho_0(x)\,\mathrm{d}x, U\rangle| > \delta\} = 0$$

for every continuous function $U:\mathbb{T}^d\to\mathbb{R}$ and every $\delta>0$. Then, the sequence of probability measures $\{Q_\mu^N,\ N>1\}$ is tight and all its limit points Q^* are concentrated *on absolutely continuous paths $\pi(t,\mathrm{d}x)=\rho(t,x)\,\mathrm{d}x$ whose density ρ is the weak solution of the equation*

$$\partial_t \rho = \sum_{k,m=1}^{d} \partial_k \{D_{k,m}(\rho)\{\partial_m \rho - 2\beta\chi(\rho)(\partial_m J * \rho)\}\},$$

$$\rho(0,\cdot) = \rho_0(\cdot) \tag{6.5}$$

and belongs to L^2 $([0,T], H_1(\mathbb{T}^d))$. Moreover, if the diffusion matrix D is locally Lipschitz continuous, then the empirical measures converge in the limit $N \to \infty$ to the unique weak solution of Eq. (6.5).

Since $\rho(x) = m(x) + 1$ the equation for the magnetization is the same as (6.5). It is not difficult to see that it can be put in the form

$$\partial_t m = \nabla \cdot \left(\Sigma \nabla \frac{\delta \mathscr{G}}{\delta m} \right) \tag{6.6}$$

with the energy functional

$$\mathscr{G}(m(r)) := \int \mathrm{d}r\, g^0(m(r)) + \frac{1}{2} \int \mathrm{d}r \int \mathrm{d}r' J(r-r') m(r) m(r'), \tag{6.7}$$

where

$$g^0(m) := -m^2 - \beta^{-1} s(m, \phi(m)) \tag{6.8}$$

and $\phi(m) = \langle \sigma^2 \rangle_m$, where $\langle \cdot \rangle_m$ is defined in (6.3). The mobility is given by the Einstein relation $\Sigma = 2\beta D(m)\chi(m)$. Moreover, the energy functional $\mathscr{G}$ is a Lyapunov functional for (6.5).

The stationary homogeneous solutions of (6.6) are given by the solutions of (2.8) with $h_1 = 0$ and $h_2 = 1$. Notice that for these values of the parameters the second equation of (2.8) determines the function $\phi(m)$ defined above as

$$\phi(m) = \frac{\cosh 2\beta m}{1 + \cosh 2\beta m}.$$

For $\beta > 3/4$ the function $g^0(m)$ has two symmetric minima $\pm m_s$ determined by the nonvanishing solutions of

$$m = \frac{\sinh 2\beta m}{1 + \sinh 2\beta m}.$$

Proof of Theorem 6.1. Following the strategy adopted in Section 4 for the Kawasaki dynamics, we divide the proof of Theorem 6.1 into three steps: Tightness, identification of the limit, and under the assumption that the diffusion matrix is locally Lipschitz continuous, the uniqueness of the hydrodynamic equation.

The proof of the tightness is essentially the same as the one given in Section 6 in Landim et al. (1998) and therefore is omitted. Notice however that in the present case of perturbed generalized simple exclusion process the invariant measures are not product measures while the proof of tightness in Kipnis et al. (1994) uses explicitly the fact that the product measures $(\nu_\alpha^N)_\alpha$ are invariant.

For the uniqueness of weak solutions of the hydrodynamic equation we need an energy estimate which states that every limit point Q^* of the sequence $\{Q_\mu^N,\ N>1\}$ is concentrated on paths whose density ρ belongs to $L^2([0,T],H_1(\mathbb{T}^d))$.

Proposition 6.2. *Let Q^* be a limit point of the sequence $\{Q_\mu^N,\ N>1\}$. Then,*

$$E_{Q^*}\left[\int_0^T \mathrm{d}s \left(\int_{\mathbb{T}^d} ||\nabla\rho(s,x)||^2\,\mathrm{d}x\right)\right] < \infty.$$

The proof of this proposition is similar to the proof of Proposition A.1.1 in Kipnis et al. (1994) and is therefore omitted. The proof of the uniqueness of weak solutions follows the same lines as in Varadhan and Yau (1997) or in Landim et al. (1998). It is here that we need the diffusion coefficient to be locally Lipschitz.

The identification of the limit is not trivial and is the main step of the proof of Theorem 6.1.

Lemma 6.3. *All limit points Q^* of the sequence $\{Q_\mu^N,\ N>1\}$ are concentrated on the paths $\rho(t,x)\,\mathrm{d}x$ whose density ρ is the weak solution of Eq.* (6.5).

Proof. Fix a function U in $\mathscr{C}^{1,2}([0,T]\times\mathbb{T}^d)$. Consider the martingales $M_t^U = M_t^{U,N}$, $N_t^U = N_t^{U,N}$ defined by

$$M_t^U = \langle \pi_t^N, U_t\rangle - \langle \pi_0^N, U_0\rangle - \int_0^t (\partial_s + N^2\mathbb{L}_\gamma^\beta)\langle \pi_s^N, U_s\rangle\,\mathrm{d}s,$$

$$N_t^U = (M_t^U)^2 - \int_0^t \{N^2\mathbb{L}_\gamma^\beta(\langle\pi_s^N, U_s\rangle)^2 - 2\langle\pi_s^N,U_s\rangle N^2\mathbb{L}_\gamma^\beta\langle\pi_s^N,U_s\rangle\}\,\mathrm{d}s.$$

In these formulae, for a continuous function U and π in $\mathscr{M}(\mathbb{T}^d)$, $\langle\pi,U\rangle$ stands for the integral of the function U with respect to the measure π.

A simple computation of the integral term of N_t^U shows that the expectation of the quadratic variation of the martingale M_t^U vanishes as $N\uparrow\infty$. Therefore, by Doob's inequality, for every $\delta>0$,

$$\lim_{N\to\infty} P_{\mu^N}\left[\sup_{0\leqslant t\leqslant T}|M_t^U|>\delta\right] = 0. \tag{6.9}$$

We now turn to the martingale M_t^U. A summation by parts permits to rewrite the integral term of the martingale M_t^U as

$$\int_0^t \langle\pi_s^N, \partial_s U_s\rangle\,\mathrm{d}s + \int_0^t \mathrm{d}s\, N^{-d+1}\sum_{k=1}^d\sum_{i\in\mathbb{T}^d}(\partial_k^N U)(s,i/N)\mathbb{W}_{i,i+e_k}^{\beta,N}(\eta_s),$$

where $\mathbb{W}_{i,i+e_k}^{\beta,N}(\eta)$ is the current over the bond $\{i,i+e_k\}$:

$$\mathbb{W}_{i,i+e_k}^{\beta,N}(\eta) = \tfrac{1}{2}\{C_\gamma^\beta(i,i+e_k;\eta) - C_\gamma^\beta(i+e_k,i;\eta)\}$$

and $(\partial_k^N U)$ is the discrete gradient defined by

$$(\partial_k^N U)(i/N) = N[U((i+e_k)/N) - U(i/N)].$$

Note that we may write the current $\mathbb{W}^{\beta,N}_{i,i+e_k}$ as the sum of the current $W_{i,i+e_k}$ of the GSEP ($\mathbb{L}^0_\gamma$) and a term coming from the perturbation:

$$\mathbb{W}^{\beta,N}_{i,i+e_k}(\eta) = W_{i,i+e_k}(\eta) + \frac{\beta}{2}N^{-1}(\partial_k J * \pi^N(\eta))(i/N)\{r_{i,i+e_k} + r_{i+e_k,i}\} + N^{-1}o_N(1),$$

where $o_N(1)$ is a random variable which is bounded in absolute value by a constant that converges to 0 as $N \uparrow \infty$ and

$$W_{i,i+e_k}(\eta) = \tfrac{1}{2}\{r_{i,i+e_k}(\eta) - r_{i+e_k,i}(\eta)\}.$$

We will often omit the dependence of $W_{i,i+e_k}$ and $\mathbb{W}^{\beta,N}_{i,i+e_k}$ on η and N.

Following the non-gradient method of Varadhan (1994) and the entropy method of Guo et al. (1988) we now replace the current $\mathbb{W}^{\beta,N}_{i,i+e_k}$ appearing in the integral term of the martingale M^U_t by a linear combination of the gradient $\{\eta(i+e_m) - \eta(i)\}$ and $\{\beta N^{-1}(\partial_m J * \pi^N(\eta))(i/N)\}$. This requires some notations. For $\ell \leqslant N$ and i in $\mathbb{T}^d$, let $\eta^\ell(x)$ stand for the mean number of particles in a cube of size $2\ell+1$ centred in i:

$$\eta^\ell(i) = \frac{1}{(2\ell+1)^d}\sum_{|j-i|\leqslant \ell}\eta(j).$$

For $k = 1,\ldots,d$, $N \geqslant 1$, $\varepsilon > 0$ and a smooth function $G : \mathbb{T}^d \to \mathbb{R}$, let

$$X^k_{N,\varepsilon}(G,\eta) = N^{-d+1}\sum_{i\in\mathbb{T}^d} G(i/N)\tau_i \mathbb{V}^{N_\varepsilon}_k(\eta),$$

where

$$\mathbb{V}^{N_\varepsilon}_k(\eta) = \mathbb{W}_{0,e_k} + \sum_{m=1}^{d} D_{k,m}(\eta^{N_\varepsilon}(0))\{[\eta^{N\varepsilon}(e_m) - \eta^{N\varepsilon}(0)]$$
$$-2\beta N^{-1}\chi(\eta^{N\varepsilon}(0))(\partial_m J * \pi^N(\eta)(0))\}.$$

The next theorem is the main step in the proof of Lemma 6.3 and therefore of the hydrodynamic limit.

Theorem 6.4. *For every smooth function $G : [0,T]\times\mathbb{T}^d \to \mathbb{R}$,*

$$\limsup_{\varepsilon\to 0}\limsup_{N\to\infty} E_{P_{\mu^N}}\left[\left|\int_0^T X^k_{N,\varepsilon}(G_s,\eta_s)\,\mathrm{d}s\right|\right] = 0$$

for $k = 1,\ldots,d$.

The proof of this theorem is postponed to the end of this section. We show now how Theorem 6.4 allows to conclude the proof of Theorem 6.1. For $1 \leqslant k,\ m \leqslant d$, denote by $d_{k,m}$ the integral of $D_{k,m}$:

$$d_{k,m}(\alpha) = \int_0^\alpha D_{k,m}(u)\,\mathrm{d}u \quad \text{for } \alpha \in [0,2[.$$

Since $D_{k,m}$ is a continuous function, by Taylor expansion,

$$d_{k,m}(\eta^{N\varepsilon}(e_m)) - d_{k,m}(\eta^{N\varepsilon}(0)) = D_{k,m}(\eta^{N\varepsilon}(0))\{\eta^{N\varepsilon}(e_m) - \eta^{N\varepsilon}(0)\} + (N\varepsilon)^{-1}o_N(1).$$

It follows therefore from Theorem 6.4 that

$$\limsup_{\varepsilon\to 0}\limsup_{N\to\infty} E_{P_{\mu^N}}\left[\left|\int_0^T \mathbb{Y}^k_{N,\varepsilon}(G_s,\eta_s)\,\mathrm{d}s\right|\right]=0,$$

where

$$\mathbb{Y}^k_{N,\varepsilon}(G,\eta)=N^{1-d}\sum_{i\in\mathrm{T}^d} G(i/N)\tau_i\mathbb{U}^{N\varepsilon}_k(\eta)$$

and

$$\mathbb{U}^{N\varepsilon}_k(\eta)=\mathbb{W}_{0,e_k}+\sum_{m=1}^d\{d_{k,m}(\eta^{N\varepsilon}(e_m))-d_{k,m}(\eta^{N\varepsilon}(0))\}$$

$$-2\beta N^{-1}\sum_{m=1}^d\{\chi(\eta^{N\varepsilon}(0))D_{k,m}(\eta^{N\varepsilon}(0))(\partial_m J*\pi^N(\eta))(0)\}.$$

A summation by parts permits to rewrite the second term of $\mathbb{Y}^i_{N,\varepsilon}(G,\eta)$ as

$$-N^{-d}\sum_{m=1}^d\sum_{i\in\mathrm{T}^d}(\partial_m G)(i/N)d_{k,m}(\eta^{N\varepsilon}(i))$$

$$-N^{-d}\sum_{m=1}^d\sum_{i\in\mathrm{T}^d}G(i/N)(2\beta\chi D_{k,m})(\eta^{N\varepsilon}(i))(\partial_m J*\pi^N(\eta))(i/N)+\mathrm{O}(N^{-1}).$$

This concludes the proof of the theorem. □

Proof of Theorem 6.4. We first introduce some notations and recall some tools used in the non-gradient methods. We denote by $\mathbb{L}^0$ the pregenerator of GSEP in infinite volume and consider the family (v_α) of invariant measures for $\mathbb{L}^0$. Let $\mathscr{C}$ be the space of cylinder functions. For each box $\Lambda\subset\mathbb{Z}^d$ and a positive integer ℓ, such that $0\leqslant\ell\leqslant 2|\Lambda|$, we denote by $v_{\Lambda,\ell}$ the canonical measure on $\{0,1,2\}^\Lambda$ with density $\ell/|\Lambda|$. For a cylinder function $g\in\mathscr{C}$, denote by Λ_g the smallest rectangle that contains the support of g and by s_g the smallest positive integer s such that $\Lambda_g\subset\Lambda_s$. Let $\mathscr{C}_0$ be the linear space of cylinder functions with mean zero with respect to all canonical invariant measures for $\mathbb{L}^0$:

$$\mathscr{C}_0=\{g\in\mathscr{C};\langle g\rangle_{\Lambda_g,\ell}=0 \text{ for all } 0\leqslant\ell\leqslant 2|\Lambda_g|\}.$$

Here $\langle g\rangle_{\Lambda_g,\ell}$ stands for the expectation of the function g with respect to the measure $v_{\Lambda_g,\ell}$.

For a positive density $0\leqslant\rho\leqslant 2$, define the semi-norm $\sqrt{\langle\langle g\rangle\rangle_\rho}$ by the central limit theorem variances

$$\langle\langle g\rangle\rangle_\rho=\lim_{\ell\to\infty}(2\ell)^{-d}\left\langle(-\mathbb{L}^0_{\Lambda_\ell})^{-1}\sum_{|i|\leqslant\ell_g}\tau_i g,\sum_{|i|\leqslant\ell_g}\tau_i g\right\rangle_{\ell,K_\ell},$$

for $g\in\mathscr{C}_0$ and a sequence of positive integers K_ℓ such that $0\leqslant K_\ell\leqslant 2(2\ell+1)^d$ and $\lim_{\ell\to\infty}K_\ell/(2\ell)^d=\rho$. In this formula $\ell_g=\ell-s_g-1$, $\Lambda_\ell=\{-\ell,\dots,\ell\}^d$ and for a box $\Lambda\subset\mathbb{Z}^d$, $\mathbb{L}^0_\Lambda$ is the restriction of $\mathbb{L}^0$ to the box Λ.

By polarization we may define from $\langle\langle\cdot\rangle\rangle_\rho$ a semi-inner product on $\mathscr{C}_0$. Moreover, if for cylinder functions g and h in $\mathscr{C}_0$ we define

$$\langle\langle g,h\rangle\rangle_{\rho,0}=\sum_i \langle g,\tau_i h\rangle_\rho$$

we obtain by the definition of $\langle\langle\cdot\rangle\rangle_\rho$ the following properties (cf. Kipnis and Landim, 1994): For all $h,g\in\mathscr{C}_0$ and for each $0\leqslant k,\ m\leqslant d$

$$\begin{aligned}
&\langle\langle g,\mathbb{L}^0 h\rangle\rangle_\rho=-\langle\langle g,h\rangle\rangle_{\rho,0},\\
&\langle\langle \eta(e_k)-\eta(0),\mathbb{L}^0 g\rangle\rangle_\rho=0,\\
&\langle\langle \eta(e_k)-\eta(0),W_{0,e_m}\rangle\rangle_\rho=-\chi(\rho)\delta_{k,m},\\
&\langle\langle W_{0,e_k},W_{0,e_m}\rangle\rangle_\rho=\tfrac{1}{2}\langle r_{0,e_1}\rangle_\rho\delta_{k,m},
\end{aligned}\tag{6.10}$$

where $\delta_{k,m}$ is the Kroenecker delta. let us denote by $\mathscr{F}$ the space of functions $F:[0,2]\times\mathbb{X}\to\mathbb{R}$ such that

(i) for each $\rho\in[0,2]$, $F(\rho,.)$ is a cylinder function with uniform support, i.e. there exists a finite set $\Lambda\subset\mathbb{Z}^d$ that contains the support of $F(\rho,\cdot)$ for all $\rho\in[0,2]$.
(ii) For each configuration η, $F(.,\eta)$ is a smooth function.

It has been proved in Kipnis et al. (1994) and Kipnis and Landim (1999) (Corollary 7.5.9 in Kipnis and Landim (1999)) that for all $0\leqslant k\leqslant d$,

$$\inf_{F\in\mathscr{F}}\ \sup_{0\leqslant\rho\leqslant 2}\left\langle\!\!\left\langle W_{0,e_k}+\sum_{m=1}^d D_{k,m}(\rho)[\eta(e_m)-\eta(0)]-\mathbb{L}^0 F(\rho,\eta)\right\rangle\!\!\right\rangle_\rho=0.$$

For each positive integer $n\geqslant 1$, let $F^{k,n}\in\mathscr{F}$ such that for all $0\leqslant\rho\leqslant 2$,

$$\left\langle\!\!\left\langle W_{0,e_k}+\sum_{m=1}^d D_{k,m}(\rho)[\eta(e_k)-\eta(0)]-\mathbb{L}^0 F^{k,n}(\rho,\eta)\right\rangle\!\!\right\rangle_\rho\leqslant\frac{1}{n}.$$

It is easy to see that for each $n\geqslant 1$ (cf. proof of (7.1.2) in Kipnis and Landim (1999))

$$\limsup_{N\to\infty}\boldsymbol{E}_{P_{\mu^N}}\left[\left|\int_0^T \mathrm{d}s N^{1-d}\sum_{i\in\mathrm{T}^d}G(s,i/N)\tau_i\mathbb{L}_\gamma^\beta F^{k,n}(\eta_s^{\varepsilon N}(0),\eta_s\right|\right]=0.$$

In particular, to prove the theorem we have to show that

$$\lim_{n\to\infty}\limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\boldsymbol{E}_{P_{\mu^N}}\left[\left|\int_0^T Z_{N,\varepsilon}^{F^{k,n}}(G_s,\eta_s)\,\mathrm{d}s\right|\right]=0,\tag{6.11}$$

where $Z_{N,\varepsilon}^{F^{k,n}}(G,\eta)$ is defined by

$$Z_{N,\varepsilon}^{F^{k,n}}(G,\eta)=X_{N,\varepsilon}^k(G,\eta)-N^{1-d}\sum_{i\in\mathrm{T}^d}G(i/N)\tau_i\mathbb{L}_\gamma^\beta F^{k,n}(\eta^{\varepsilon N}(0),\eta).$$

On the other hand, since $F^{k,n}$ is a cylinder function, it is easy to see by Lemma 6.9 that for all $i \in \mathbb{T}^d$

$$\tau_i \mathbb{L}_\gamma^\beta F^{n,k} = N^{-1} o_N(1) + \tau_i \mathbb{L}_\gamma^0 F^{n,k} + N^{-1}\frac{\beta}{2}\sum_{m=1}^d (\partial_m J * \pi^N(i/N))\tau_i \left\{ \sum_{j\in \mathrm{T}^d} \{\nabla_{j,j+e_m} F^{n,k} - \nabla_{j+e_m,j} F^{n,k}\} \right\}.$$

We now decompose $Z_{N,\varepsilon}^{F^{k,n}}$ into two parts, $Z_{N,\varepsilon}^{F^{k,n},1}$ and $Z_{N,\varepsilon}^{F^{k,n},2}$:

$$Z_{N,\varepsilon}^{F^{k,n}} = Z_{N,\varepsilon}^{F^{k,n},1} + Z_{N,\varepsilon}^{F^{k,n},2} + N^{-1} o_N(1),$$

where

$$Z_{N,\varepsilon}^{F^{k,n},1} = N^{1-d} \sum_{i\in\mathbb{T}^d} G(i/N)\tau_i \left\{ W_{0,e_k} + \sum_{m=1}^d D_{k,m}(\eta^{N\varepsilon}(0))[\eta^{N\varepsilon}(e_m) - \eta^{N\varepsilon}(0)] \right\} - N^{1-d}\sum_{i\in\mathbb{T}^d} G(i/N)\tau_i \mathbb{L}_\gamma^0 F^{n,k}(\eta^{\varepsilon N}(0),\eta),$$

$$Z_{N,\varepsilon}^{F^{k,n},2} = \beta N^{-d}\sum_{i\in\mathbb{T}^d} G(i/N)\tau_i \left\{ (\partial_k J * \pi^N(\eta)(0))\left(\frac{r_{0,e_k} + r_{e_k,0}}{2}\right) - 2\sum_{m=1}^d (\chi D_{k,m})(\eta^{N\varepsilon}(0))(\partial_m J * \pi^N(\eta)(0)) \right\}$$
$$- \beta N^{-d}\sum_{i\in\mathbb{T}^d} G(i/N)\tau_i \left\{ \sum_{m=1}^d (\partial_m J * \pi^N(\eta)(0)) \times \left\{ \sum_{j\in\mathbb{T}^d} (1/2)(\nabla_{j,j+e_m} F^{n,k} - \nabla_{j+e_m,j} F^{n,k}) \right\} \right\}.$$

Here $(\chi D_{k,m})$ represents the product function $(\chi D_{k,m})(\alpha) = \chi(\alpha) D_{k,m}(\alpha)$. To conclude the proof of the theorem it is enough to prove the following lemmas:

Lemma 6.5. *For each* $0 \leqslant k \leqslant d$,

$$\lim_{n\to\infty}\limsup_{\varepsilon\to 0}\limsup_{N\to\infty} \boldsymbol{E}_{P_{\mu^N}}\left[\left|\int_0^T Z_{N,\varepsilon}^{F^{k,n},1}(G_s,\eta_s)\,\mathrm{d}s\right|\right] = 0. \tag{6.12}$$

Lemma 6.6. *For each* $0 \leqslant k \leqslant d$,

$$\lim_{n\to\infty}\limsup_{\varepsilon\to 0}\limsup_{N\to\infty} \boldsymbol{E}_{P_{\mu^N}}\left[\left|\int_0^T Z_{N,\varepsilon}^{F^{k,n},2}(G_s,\eta_s)\,\mathrm{d}s\right|\right] = 0. \tag{6.13}$$

Proof of Lemma 6.5. Fix $0 \leqslant k \leqslant d$ and denote $Z_{N,\varepsilon}^{F^{n,k},1}$ by $Z_{N,\varepsilon}^1$. Since the entropy $s(P_{\mu^N}|P_{\nu_\alpha^N})$ of P_{μ^N} with respect to $P_{\nu_\alpha^N}$ is equal to $s(\mu^N|\nu_\alpha^N)$ which is bounded by

CN^d for some positive constant C, by the entropy inequality for any positive A

$$\begin{aligned}&\boldsymbol{E}_{P_{\mu^N}}\left[\left|\int_0^t Z^1_{N,\varepsilon}(G_s,\eta_s)\,\mathrm{d}s\right|\right]\\&\quad\leqslant \frac{C}{A}+\frac{1}{AN^d}\log \boldsymbol{E}_{P_{\nu^N_\alpha}}\left[\exp\left(AN^d\left|\int_0^t Z^1_{N,\varepsilon}(G_s,\eta_s)\,\mathrm{d}s\right|\right)\right].\end{aligned}$$

Since $\mathrm{e}^{|x|}\leqslant \mathrm{e}^x+\mathrm{e}^{-x}$ and

$$\limsup N^{-d}\log\{a_N+b_N\}\leqslant \max\{\limsup N^{-d}\log a_N, \limsup N^{-d}\log b_N\},$$

the absolute value appearing in the exponent on the right-hand side of the last inequality can be eliminated. Indeed, by the Feynman–Kac formula, we have

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\frac{1}{AN^d}\log \boldsymbol{E}_{P_{\nu^N_\alpha}}\left[\exp\left(AN^d\left|\int_0^t Z^1_{N,\varepsilon}(G_s,\eta_s)\,\mathrm{d}s\right|\right)\right]\\&\quad\leqslant \limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\frac{1}{AN^d}\int_0^t(\lambda_{N,\varepsilon}(G_s)+\lambda_{N,\varepsilon}(-G_s))\,\mathrm{d}s,\end{aligned}$$

where $\lambda_{N,\varepsilon}(\pm G_s)$ is the largest eigenvalue of the reversible operator

$$\frac{N^2}{2}(\mathbb{L}^\beta_\gamma+\mathbb{L}^{\beta,*}_\gamma)+AN^dZ^1_{N,\varepsilon}(\pm G_s,\eta)$$

given by the variational formula

$$\sup_{\substack{f\geqslant 0\\ \int f d\nu^N_\alpha=1}}\left\{AN^d\int Z^1_{N,\varepsilon}(\pm G_s,\eta)f(\eta)\,\mathrm{d}\nu^N_\alpha(\eta)+N^2\langle\sqrt{f},\mathbb{L}^\beta_\gamma\sqrt{f}\rangle_\alpha\right\}.$$

Here, $\mathbb{L}^{\beta,*}_\gamma$ stands for the adjoint operator of $\mathbb{L}^\beta_\gamma$ in $L^2(\nu^N_\alpha)$. From Lemma 6.8 below we have, for all positive A

$$\begin{aligned}\frac{1}{AN^d}\int_0^t\lambda_{N,\varepsilon}(\pm G_s)\,\mathrm{d}s\leqslant \frac{C_1}{A}t+\int_0^t\mathrm{d}s\left\{\sup_{\substack{f\geqslant 0\\ \int f d\nu^N_\alpha=1}}\left\{\int Z^1_{N,\varepsilon}(\pm G_s,\eta)f(\eta)\,\mathrm{d}\nu^N_\alpha(\eta)\right.\right.\\ \left.\left.+\frac{N^{2-d}}{A}\langle\sqrt{f},\mathbb{L}^0_\gamma\sqrt{f}\rangle_\alpha\right\}\right\}.\end{aligned}$$

It has been proven in Kipnis et al. (1994) and Kipnis and Landim (1999) that for any positive A and any smooth function $G\in\mathscr{C}^{1,1}([0,T]\times\mathbb{T}^d)$ and any α

$$\begin{aligned}\lim_{n\to\infty}\limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\int_0^t\mathrm{d}s\left\{\sup_{\substack{f\geqslant 0\\ \int f d\nu^N_\alpha=1}}\left\{\int Z^1_{N,\varepsilon}(G_s,\eta)f(\eta)\,\mathrm{d}\nu^N_\alpha(\eta)\right.\right.\\ \left.\left.+AN^{2-d}\langle\sqrt{f},\mathbb{L}^0_\gamma\sqrt{f}\rangle_\alpha\right\}\right\}=0.\end{aligned}$$

This concludes the proof of Lemma 6.5. □

Proof of Lemma 6.6. Fix $0 \leqslant k \leqslant d$. To simplify the notation, for $0 \leqslant m \leqslant d$ denote by ψ^1 and $\psi^{2,m}$ the cylinder functions $\Psi^1 = r_{0,e_1}$ and

$$\Psi_k^{2,m} = \left\{ \sum_{j \in \mathrm{T}^d} (1/2)(\nabla_{j,j+e_m} F^{n,k} - \nabla_{j+e_m,j} F^{n,k}) \right\}.$$

For a cylinder function Ψ and a positive density $0 \leqslant \rho \leqslant 2$, we shall denote by $\tilde{\Psi}(\rho)$ its expectation with respect to the measure ν_ρ^N.

Observe that for each $0 \leqslant m \leqslant d$ and all density $0 \leqslant \rho \leqslant 2$, $\tilde{\Psi}_k^{2,m}(\rho)$ can be rewritten as

$$\begin{aligned} \tilde{\Psi}_k^{2,m}(\rho) &= -\langle 2W_{0,e_m}, F^{n,k} \rangle_{\rho,0} \\ &= \langle\langle 2W_{0,e_m}, \mathbb{L}^0 F^{n,k} \rangle\rangle_\rho. \end{aligned}$$

In particular, from Lemma 6.8 and using one and two blocks estimates, we obtain

$$\left| \int_0^T Z_{N,\varepsilon}^{F^{k,n},2}(G_s, \eta_s)\, \mathrm{d}s \right| = \left| \int_0^T \tilde{Z}_{N,\varepsilon}^{F^{k,n},2}(G_s, \eta_s)\, \mathrm{d}s \right| + r_{N,\varepsilon},$$

where $r_{N,\varepsilon}$ is a random variable such that its expectation with respect to P_{μ^N} converges to 0 as $N \uparrow \infty$ and $\varepsilon \downarrow 0$ and

$$\begin{aligned} \tilde{Z}_{N,\varepsilon}^{F^{k,n},2}(G,\eta) = \beta N^{-d} \sum_{i \in \mathrm{T}^d} G(i/N) \Bigg\{ & \sum_{m=1}^{d} (\partial_m J * \pi^N(\eta)(i/N)) \{ \tilde{\Psi}^1(\eta^{N\varepsilon}(i)) \delta_{k,m} \\ & - 2(\chi D_{k,m})(\eta^{N\varepsilon}(i)) - \tilde{\Psi}_k^{2,m}(\eta^{N\varepsilon}(i)) \Bigg\}. \end{aligned}$$

To conclude the proof we just have to apply Lemma 6.7 below. □

Lemma 6.7. *There exists a positive constant C_0 such that for each $0 \leqslant m \leqslant d$*

$$\sup_{0 \leqslant \rho \leqslant 2} |\langle r_{0,e_1} \rangle_\rho \delta_{k,m} - 2\chi(\rho) D_{k,m}(\rho) - 2\langle\langle W_{0,e_m}, \mathbb{L}^0 F^{k,n} \rangle\rangle_\rho | \leqslant \frac{C_0}{\sqrt{n}}.$$

Proof. Using (6.10), it is easy to show that

$$\begin{aligned} &|\langle r_{0,e_1} \rangle_\rho \delta_{k,m} - 2\chi(\rho) D_{k,m}(\rho) - 2\langle\langle W_{0,e_m}, \mathbb{L}^0 F^{k,n} \rangle\rangle_\rho | \\ &\quad = \left| \left\langle \left\langle 2W_{0,m}, W_{0,e_k} + \sum_{r=1}^{d} D_{k,r}(\rho)[\eta(e_r) - \eta(0)] - \mathbb{L}^0 F^{k,n} \right\rangle \right\rangle_\rho \right|. \end{aligned}$$

By Schwartz inequality the right-hand side of the last expression is bounded by

$$\sqrt{\langle\langle 2W_{0,m} \rangle\rangle_\rho} \times \left(\left\langle \left\langle W_{0,e_k} + \sum_{r=1}^{d} D_{k,r}(\rho)[\eta(e_r) - \eta(0)] - \mathbb{L}^0 F^{k,n} \right\rangle \right\rangle_\rho \right)^{1/2}$$

which is bounded by $C_0/\sqrt{n}$ for some positive constant C_0. □

We now prove some estimates on the Dirichlet forms for the process at $\beta \neq 0$ needed in the proof of Theorem 6.4 and in the proof of one and two blocks estimates.

Lemma 6.8. *There exists a positive constant C_1 such that for every probability density f with respect to ν_α^N*

$$N^2\langle\sqrt{f},\mathbb{L}_\gamma^\beta\sqrt{f}\rangle_{\nu_\alpha^N}\leqslant\frac{1}{2}N^2\langle\sqrt{f},\mathbb{L}_\gamma^0\sqrt{f}\rangle_{\nu_\alpha^N}+C_1N^d.$$

Proof. Fix a probability density f. A simple computation shows that

$$\begin{aligned}N^2\langle\sqrt{f},\mathbb{L}_\gamma^\beta\sqrt{f}\rangle_{\nu_\alpha^N}&=N^2\langle\sqrt{f},\mathbb{L}_\gamma^0\sqrt{f}\rangle_{\nu_\alpha^N}\\&\quad+\frac{1}{2}\sum_{\substack{i,j\in\mathbb{T}^d\\|i-j|=1}}\int r_{i,j}[1-\Phi\{\beta(\nabla^{i,j}H_N)(\eta)\}]\sqrt{f}[\nabla_{i,j}\sqrt{f}](\eta)\,\mathrm{d}\nu_\alpha^N.\end{aligned}\tag{6.14}$$

Recall that, by Lemma 6.9 below,

$$|1-\Phi\{\beta(\nabla^{i,j}H_N)(\eta)\}|\leqslant C_2N^{-1}$$

for some positive constant C_2. To conclude the proof of the lemma it remains to bound the second term on the right-hand side of (6.14) by

$$-(N^2/2)\langle\sqrt{f},\mathbb{L}_\gamma^0\sqrt{f}\rangle_{\nu_\alpha^N}+C_2N^d$$

using the inequality $2xy\leqslant ax^2+a^{-1}y^2$ for all $a>0$ and the fact that f is a density with respect to ν_α^N. $\square$

Lemma 6.9. *For all $i\in\mathbb{T}^d$, unit vector $e\in\mathbb{T}^d$ and $\eta\in\mathbb{X}_N$*

$$C_\gamma^\beta(i,i+e;\eta)=1+\left\{\beta N^{-d-1}\sum_{j\in\mathrm{T}^d}(e\cdot\nabla J)((i-j)/N)\eta(j)\right\}+\mathrm{O}(N^2).$$

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Appendix

Proof of (5.1). We have

$$\begin{aligned}\sum_{k=1}^d|\partial_k h_{t+\varepsilon-s}(x)|&=(2\pi(t+\varepsilon-s))^{-d/2}\exp\left\{-\frac{1}{8(t+\varepsilon-s)}||x||^2\right\}\times\frac{1}{\sqrt{t+\varepsilon-s}}\\&\quad\times\frac{\sum_{k=1}^d|x_k|}{\sqrt{t+\varepsilon-s}}\exp\left\{-\frac{1}{8(t+\varepsilon-s)}||x||^2\right\},\end{aligned}$$

where for $x \in \mathbb{R}^d$, $||x|| = \sqrt{\sum_{k=1}^d (x_k)^2}$. Since $\sum_{k=1}^d |x_i| \leqslant d||x||$ and $\alpha \exp(-\alpha^2) \leqslant A_1$ for all $\alpha \in \mathbb{R}$ and for some positive constant A_1. We have that

$$\sum_{k=1}^d |\partial_k h_{t+\varepsilon-s}(x)| \leqslant A_2 h_{2(t+\varepsilon-s)}(x) \frac{1}{\sqrt{t+\varepsilon-s}}$$

for some positive constant A_2 that depends on d. On the other hand, we have

$$\sum_{k=1}^d \partial_k \mathscr{H}_{t,\varepsilon}^f = f * \left(\sum_{k=1}^d \partial_k h_{t+\varepsilon-s} \right)$$

and therefore

$$\begin{aligned}
&\sum_{k=1}^d \int_0^t |\langle \partial_k \mathscr{H}_{t,\varepsilon}^f(s,.)\rangle| \,\mathrm{d}s \\
&\quad \leqslant \int_0^t \mathrm{d}s \left\{ \int_{\mathbb{R}^d} \mathrm{d}x \left[\int_{\mathbb{R}^d} |f(y)| \sum_{k=1}^d |\partial_k h_{t+\varepsilon-s}(y-x)| \,\mathrm{d}y \right] \right\} \\
&\quad \leqslant A_2 \int_0^t \mathrm{d}s \left\{ \int_{\mathbb{R}^d} \mathrm{d}y |f(y)| \left[\int_{\mathbb{R}^d} h_{2(t+\varepsilon-s)}(y-x) \,\mathrm{d}x \right] \times \frac{1}{\sqrt{t+\varepsilon-s}} \right\} \\
&\quad \leqslant A_2 ||f||_1 \int_0^t \frac{1}{\sqrt{t+\varepsilon-s}} \mathrm{d}s \\
&\quad = A_2 ||f||_1 |\{\sqrt{t+\varepsilon} - \sqrt{\varepsilon}\}. \qquad \square
\end{aligned}$$

Proof of (5.3). For all $\tau \in [0,t]$, $W(\tau,.) \in L^\infty(\mathbb{R}^d)$ and $||W(\tau,.)||_\infty \leqslant 2C_3\sqrt{t}R(t)$.

First, by (5.2), for any open set U of $\mathbb{R}^d$ with finite Lebesgue measure $\lambda(U)$, we have for all $0 \leqslant \tau \leqslant t$,

$$\int_U W(\tau,x) \,\mathrm{d}x \leqslant 2C_3\sqrt{t}R(t)\lambda(U).$$

Fix $0 < \delta < 1$. For any open set U of $\mathbb{R}^d$ with finite Lebesgue measure and for $0 \leqslant \tau \leqslant t$ let

$$B_{\delta,\tau}^U = \{x \in U\colon W(\tau,x) > 2C_3\sqrt{t}R(t)(1+\delta)\}.$$

Suppose that $\lambda(B_{\delta,\tau}^U) > 0$. Then there exists an open set V, such that, $B_{\delta,\tau}^U \subset V$ and $\lambda(V \setminus B_{\delta,\tau}^U) \leqslant \lambda(V)^{\delta/2}$ and we have

$$\begin{aligned}
\lambda(V)(2C_3\sqrt{t}R(t)) &< \lambda(V)(2C_3\sqrt{t}R(t))(1+\delta)(1-\delta/2) \\
&= (2C_3\sqrt{t}R(t))(1+\delta)(\lambda(V) - \lambda(V)\delta/2) \\
&\leqslant (2C_3\sqrt{t}R(t))(1+\delta)(\lambda(V) - \lambda(V \setminus B_{\delta,\tau}^U)) \\
&= (2C_3\sqrt{t}R(t))(1+\delta)\lambda(B_{\delta,\tau}^U) \\
&< \int_{B_{\delta,\tau}^U} W(\tau,x) \,\mathrm{d}x
\end{aligned}$$

$$\leqslant \int_V W(\tau,x)\,\mathrm{d}x$$
$$\leqslant (2C_3\sqrt{t}R(t))\lambda(V),$$

which leads to a contradiction.

By the arbitrariness of $0 < \delta < 1$ we obtain that if U is any open set of $\mathbb{R}^d$ with $\lambda(U) < \infty$,

$$\lambda(\{x \in U\colon W(\tau,x) > 2C_3\sqrt{t}R(t)\}) = 0.$$

This implies that

$$W(\tau,x) \leqslant 2C_3\sqrt{t}R(t) \quad \text{a.e. in } \mathbb{R}^d. \qquad \square$$

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